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Journal of Computational Physics 204 (2005) 404–429
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Derivation of high-order compact finite difference schemes for non-uniform grid using polynomial interpolation

Ratnesh K. Shukla, Xiaolin Zhong *

Mechanical and Aerospace Engineering Department, University of California, 46-147C, Engineering IV Building, 420 Westwood Plaza, Los Angeles, CA 90095-1597, USA

Received 21 November 2003; received in revised form 13 October 2004; accepted 13 October 2004

Abstract

In this paper simple polynomial interpolation is used to derive arbitrarily high-order compact schemes for the first derivative and tridiagonal compact schemes for the second derivative (consisting of three second derivative nodes in the interior and two on the boundary) on non-uniform grids. Boundary and near boundary schemes of the same order as the interior are also developed using polynomial interpolation and for a general compact scheme on a non-uniform grid it is shown that polynomial interpolation is more efficient than the conventional method of undetermined coefficients for finding coefficients of the scheme. The high-order non-uniform schemes along with boundary closure of up to 14th order thus obtained are shown to be stable on a non-uniform grid with appropriate stretching so that more grid points are clustered near the boundary. The stability and resolution properties of the high-order non-uniform grid schemes are studied and the results of three numerical tests on stability and accuracy properties are also presented.

Keywords: Compact finite difference schemes; Non-uniform grid; Polynomial Interpolation

1. Introduction

Compact high-order finite difference schemes which consider as unknowns at each discretization point not only the value of the function but also those of its first or higher derivatives have been extensively studied and widely used to compute problems involving incompressible, compressible and hypersonic flows [27–33,35–38], computational aeroacoustics [6,21] and several other practical applications [39–

* Corresponding author. Tel.: +1 310 825 2905.
E-mail addresses: ratnesh@seas.ucla.edu (R.K. Shukla), xiaolin@seas.ucla.edu (X. Zhong).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2004.10.014

44]. Compared to explicit finite difference schemes, these schemes are implicit and give a higher order of accuracy for the same number of grid points and also provide high resolution characteristics. In addition, compact schemes are more flexible in terms of application to complex geometries and boundary conditions when compared to spectral methods. Lele [3] went through extensive analysis of compact schemes and applied them for solution of compressible and incompressible flow problems. Deng and Maekawa [23] and Deng and Zhang [22] developed non-linear and weighted non-linear compact schemes, respectively, for capturing discontinuities. Prefactored small stencil compact differencing schemes which split the implicit compact operator matrix into upper and lower matrices and are simpler to implement were developed and analyzed in [24–26]. In [13] Mahesh presented and analyzed combined compact uniform grid finite difference schemes which evaluate both the first and the second derivative simultaneously. Goedheer and Potters [12] derived a fourth-order non-uniform combined compact finite difference scheme using truncated Taylor series and used it to solve a model transport problem in one-dimension. In [15] Chu and Fan derived sixth-order and eighth-order three point combined compact finite difference schemes on a uniform grid and also mentioned the existence of local and global Hermite polynomials whose coefficients were calculated using Taylor series expansion. The same authors later extended the derivation of sixth-order combined compact uniform grid scheme to a non-uniform grid scheme in [14]. Ge and Zhang [43] used a coordinate transformation from a non-uniform to a uniform grid to solve a two-dimensional steady state convection–diffusion equation using a fourth-order nine point uniform grid compact scheme. Finite volume compact schemes have been developed and applied to solve flow problems in [17–20].

Traditionally compact schemes have been derived for a uniform grid. Recently, a number of authors have investigated applications of high-order compact schemes to non-uniform grids. The usual approach is to use a mapping from non-uniform grid to a uniform grid and apply the compact schemes for uniform grids directly on the mapped coordinate. Gamet et al. [5] adapted the compact schemes originally developed for uniform meshes to non-uniform meshes using metrices of the grid. Cheong and Lee [6] developed GODRP schemes which are designed to have locally the same dispersion relation as the partial differential equation and have optimized dissipation characteristics at the non-uniform cartesian or curvilinear grids. Visbal and Gaitonde [7] have shown that application of high-order compact schemes to non-smooth and time varying grids using Jacobian transformation leads to spurious oscillations, unless a proper filtering scheme is used. Kwok et al. [8] compared the resolution properties of B-spline and compact finite difference schemes on non-uniform grids by transforming the non-uniform grid to a uniform grid.

As outlined before, the popular method for application of compact schemes on non-uniform grids is to use a Jacobian transformation from the uniform grid to the non-uniform grid. Carpenter et al. [9] showed that for a sixth-order interior compact scheme on a uniform grid only a third-order boundary scheme can be used without introducing instability, which results in a globally fourth-order scheme. They also developed asymptotically stable schemes by removing the constraint of optimal accuracy by increasing the stencil width of compact schemes thus enabling several parameter boundary closures. Abarbanel et al. [10,11] developed a methodology for construction of high-order compact schemes on uniform grids for hyperbolic initial and boundary value problems by generalizing the procedure proposed by [9]. However, construction of stable schemes using this method is a difficult task and has been done only for up to sixth-order schemes. Recently, Zhong and Mahidhar [4] presented high-order (up to 12th order) non-uniform stable finite difference schemes with boundary closure of same order as the interior.

The conventional method of deriving compact difference schemes using a truncated Taylor series and determining the coefficients of the scheme based on the desired accuracy becomes cumbersome in case of high-order schemes on non-uniform grids since it requires computation of different sets of coefficients for each grid point as the grid spacing is no longer uniform and is also prone to numerical errors involved in matrix inversion. Thus it will be useful if coefficients of the compact scheme could be obtained through a direct derivation. Hence, our objective in this paper is twofold:

- First we present a simpler way of deriving the compact schemes by use of polynomial interpolation and for an illustration describe the procedure for two test cases. For the case of first derivative we use Hermite–Birkhoff interpolation [1] to obtain an explicit form of compact difference schemes. For the case of second order or higher derivative, the polynomial interpolation problem to be considered is a special case of the general Birkhoff Interpolation problem [1]. Suzuki [2] derived a method of constructing the interpolation polynomial for the (0,q) Birkhoff Interpolation problem (0 represents function values and q represents qth derivative) from the polynomials of (0,1) Hermite–Birkhoff interpolation polynomials and also gave condition for existence of these polynomials. However, in this paper we use a different simplified approach in order to derive the explicit analytical form for the general second-order tridiagonal scheme. The schemes constructed by using polynomial interpolation correspond to the Padé schemes, in that they have the highest accuracy within the family of compact schemes that can be constructed on a specified computational stencil.
- Next we evaluate the stability of high-order (up to 14th order) non-uniform compact schemes with same order of boundary closure as the interior for a grid with grid points clustered at the boundary following the approach of [4] for the case of high-order finite difference schemes on non-uniform grids and determine the amount of grid stretching required to obtain stable schemes. The schemes are then tested by computing solution of linear one-dimensional and two-dimensional wave equation with time oscillatory boundary conditions and a two-dimensional linear convection–diffusion equation.

2. Derivation of compact schemes by direct polynomial interpolation

2.1. Polynomial interpolation

A general compact finite difference scheme for one dimension along x centered at x_i has the form $u_i^{(p)}+\sum_{j\in I_n}a_ju_j^{(p)}=\sum_{j\in I_n}b_ju_j+\sum_{j\in I_m}b_ju_j$, where u_j are the function values given at set of points $x_j\in I_n\cup I_m$ and $u_j^{(p)}$ are the values of the pth derivative of the function given at set of points $x_j\in I_n$ and the point x_i is included in the set I_m. In the following sections we shall consider the problem of finding out the coefficients a_j and b_j in the compact scheme for two values of p, 1 and 2, corresponding to first- and second-order compact schemes, respectively, using univariate (0,p) interpolation polynomial, where (0,p) refers to the fact that polynomial interpolates through an arbitrarily distributed unique set of points on which either the function value or the value of pth derivative of the function or both have been specified.

2.2. (0,1) Interpolation

Consider a set of n points $\mathbf{I_n}$ on which values of the function and its first derivative have been specified and another set of m points $\mathbf{I_m}$ on which only function values have been specified. The independent variable representing the points is x_i, i being the index of the node and the function values are given by $u_i = u(x_i)$ and the first derivative is given by $u_i'=u'(x_i)$. Then a polynomial $u(x)$ of degree $\leqslant 2n + m - 1$ that assumes the values $u_i = u(x_i)$, $i \in I_n \cup I_m$ and $u_i'=u'(x_i), i\in I_n$ is of the form

$$u(x)=\sum_{i\in I_n}u_i\rho_i(x)+\sum_{i\in I_n}u_i'q_i(x)+\sum_{i\in I_m}u_ir_i(x), \tag{1}$$

where the polynomials $\rho_i(x)$, $q_i(x)$ and $r_i(x)$ satisfy the following conditions:

$$\begin{aligned}
&\rho_i(x_j) = \delta_{ij} \quad \forall i \in I_n,\ \forall j \in I_n \cup I_m, \qquad \rho_i'(x_j) = 0 \quad \forall i \in I_n,\ \forall j \in I_n,\\
&q_i(x_j) = 0 \quad \forall i \in I_n,\ \forall j \in I_n \cup I_m, \qquad q_i'(x_j) = \delta_{ij} \quad \forall i \in I_n,\ \forall j \in I_n,\\
&r_i(x_j) = \delta_{ij} \quad \forall i \in I_m,\ \forall j \in I_n \cup I_m, \qquad r_i'(x_j) = 0 \quad \forall i \in I_m,\ \forall j \in I_n,
\end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta. The conditions (2) suggest following form of the polynomials $\rho_i(x)$:

$$\rho_i(x) = \frac{\prod_m(x)}{\prod_m(x_i)} l_i^n(x)\left\{1 + \sum_{r=1}^{n} A_r(x - x_i)^r\right\}, \quad i \in I_n, \tag{3}$$

where $l_i^n(x)$ (lagrange polynomials on I_n) and $\prod_m(x)$ are defined as

$$l_i^n(x) = \frac{\prod_{j\in I_n \neq i}(x - x_j)}{\prod_{j\in I_n \neq i}(x_i - x_j)} \quad \text{and} \quad \prod_m(x) = \prod_{j\in I_m}(x - x_j).$$

Using (2) and (3) it is easy to find that

$$\rho_i(x) = \frac{\prod_m(x)}{\prod_m(x_i)}\left\{l_i^n(x)\right\}^2\left[1 - \left\{2l_i^{n'}(x_i) + \frac{\prod_m'(x_i)}{\prod_m(x_i)}\right\}(x - x_i)\right] \quad \forall i \in I_n. \tag{4}$$

A similar analysis for $q_i(x)$ and $r_i(x)$ gives

$$q_i(x) = \frac{(x - x_i)\prod_m(x)}{\prod_m(x_i)}\left\{l_i^n(x)\right\}^2 \quad \forall i \in I_n, \qquad r_i(x) = \left\{\frac{\prod_n(x)}{\prod_n(x_i)}\right\}^2 l_i^m(x) \quad \forall i \in I_m, \tag{5}$$

where $l_i^m(x)$ are lagrange polynomials on I_m and $\prod_n(x) = \prod_{j\in I_n}(x - x_j)$.

2.3. (0,2) Interpolation

Consider a set of n points $\mathbf{I_n}$ on which values of the function and its second derivative have been specified and another set of m points $\mathbf{I_m}$ on which only function values have been specified. The function values are given by $u_i = u(x_i)$ and the second derivative is given by $u_i'' = u''(x_i)$.Then a polynomial $u(x)$ of degree $\leqslant 2n + m - 1$ that assumes the values $u_i = u(x_i)$, $i \in I_n \cup I_m$ and $u_i'' = u''(x_i),\ i \in I_n$ is of the form

$$u(x) = \sum_{i\in I_n} u_i\rho_i(x) + \sum_{i\in I_n} u_i'' q_i(x) + \sum_{i\in I_m} u_i r_i(x), \tag{6}$$

where the polynomials $\rho_i(x)$, $q_i(x)$ and $r_i(x)$ satisfy the following conditions:

$$\begin{aligned}
&\rho_i(x_j) = \delta_{ij} \quad \forall i \in I_n,\ \forall j \in I_n \cup I_m, \qquad \rho_i''(x_j) = 0 \quad \forall i \in I_n,\ \forall j \in I_n,\\
&q_i(x_j) = 0 \quad \forall i \in I_n,\ j \in I_n \cup I_m, \qquad q_i''(x_j) = \delta_{ij} \quad \forall i \in I_n,\ \forall j \in I_n,\\
&r_i(x_j) = \delta_{ij} \quad \forall i \in I_m,\ \forall j \in I_n \cup I_m, \qquad r_i''(x_j) = 0 \quad \forall i \in I_m,\ \forall j \in I_n.
\end{aligned} \tag{7}$$

As before we have the following form of the polynomials $\rho_i(x)$:

$$\rho_i(x) = \frac{\prod_m(x)}{\prod_m(x_i)} l_i^n(x)\left(1 + \sum_{r=1}^{n} A_r(x - x_i)^r\right), \quad i \in I_n. \tag{8}$$

Differentiating this relation twice, putting $x = x_j$ and using (7) we have

$$2A_2 + 2A_1\left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}'\bigg|_{x=x_i} + \left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}''\bigg|_{x=x_i} = 0, \quad j = i,$$

$$\sum_{r=1}^{n} A_r\left[(x_j - x_i)^r\left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}''\bigg|_{x=x_j} + 2r(x_j - x_i)^{r-1}\left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}'\bigg|_{x=x_j}\right] + \left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}''\bigg|_{x=x_j}$$
$$= 0 \quad \forall j \neq i, \;\; i \in I_n \text{ and } j \in I_n, \tag{9}$$

which gives n equations in n unknowns $A_1, A_2, \ldots, A_n$. Similarly using (7) if $q_i(x)$ is assumed to be of the form

$$q_i(x) = \frac{\prod_m(x)}{\prod_m(x_i)} l_i^n(x)\left(\sum_{r=1}^{n} B_r(x - x_i)^r\right), \quad i \in I_n \tag{10}$$

we have for the coefficients B_i,

$$2B_2 + 2B_1\left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}'\bigg|_{x=x_i} = 1, \quad j = i,$$

$$\sum_{r=1}^{n} B_r\left[(x_j - x_i)^r\left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}''\bigg|_{x=x_j} + 2r(x_j - x_i)^{r-1}\left\{\frac{\prod_m(x)l_i^n(x)}{\prod_m(x_i)}\right\}'\bigg|_{x=x_j}\right] = 0$$
$$\forall j \neq i, \;\; i \in I_n \text{ and } j \in I_n. \tag{11}$$

Also from (7) if form of $r_i(x)$ is

$$r_i(x) = \frac{\prod_n(x)}{\prod_n(x_i)} l_i^m(x)\left(1 + \sum_{r=1}^{n} C_r(x - x_i)^r\right), \quad i \in I_m \tag{12}$$

we have for the coefficients C_i

$$\sum_{r=1}^{n} C_r\left[(x_j - x_i)^r\left\{\frac{\prod_n(x)l_i^m(x)}{\prod_n(x_i)}\right\}''\bigg|_{x=x_j} + 2r(x_j - x_i)^{r-1}\left\{\frac{\prod_n(x)l_i^m(x)}{\prod_n(x_i)}\right\}'\bigg|_{x=x_j}\right] + \left\{\frac{\prod_n(x)l_i^m(x)}{\prod_n(x_i)}\right\}''\bigg|_{x=x_j}$$
$$= 0, \quad \forall j \in I_n, \;\; i \in I_m. \tag{13}$$

2.4. Scheme for first derivative

The general compact scheme for the first derivative centered at x_i can be derived from the interpolation polynomial given by (1), (4) and (5) by taking the first derivative at $x = x_i$ as

$$
\begin{aligned}
&u_i' + \sum_{j \in I_n} a_j u_j' = b_i u_i + \sum_{j \in I_m \neq i} b_j u_j + \sum_{j \in I_n} c_j u_j, \\
&b_i = \left\{ l_i^{m'}(x_i) + 2\frac{\prod_n'(x_i)}{\prod_n(x_i)} \right\}, \\
&b_j = \left\{ \frac{\prod_n(x_i)}{\prod_n(x_j)} \right\}^2 l_j^{m'}(x_i), \\
&a_j = \frac{(x_j - x_i)\prod_m'(x_i)}{\prod_m(x_j)} \{ l_j^n(x_i) \}^2, \\
&c_j = \frac{\prod_m'(x_i)}{\prod_m(x_j)} \{ l_j^n(x_i) \}^2 \left[1 - \left\{ 2 l_j^{n'}(x_j) + \frac{\prod_m'(x_j)}{\prod_m(x_j)} \right\} (x_i - x_j) \right].
\end{aligned}
\tag{14}
$$

Using proper choice of sets of points, I_m and I_n, arbitrary order scheme can be constructed for the interior or the boundary on a non-uniform grid. Also if I_m has m points and I_n has n points then the order of the scheme will be $2n + m - 1$. For example fourth-order accurate first derivative tridiagonal compact schemes for interior and boundary points for a uniform and a non-uniform grid with the distribution of nodes given by $x_i = x_1 + h(i - 1)$, $i = 1,2,\ldots,N$ and $x_i = x_1 + h(i - 1)^2$, $i = 1,2,\ldots,N$, respectively, are presented in Table 1 along with the choice of sets I_n and I_m needed to derive them. Additional examples of first derivative compact finite difference schemes on uniform and non-uniform grid are listed in Table B.1 in Appendix B.

2.5. Scheme for second derivative

The method discussed in Section 2.3 can be used to derive the form of general tridiagonal compact scheme for the second derivative for an arbitrary grid point distribution. For deriving the interior scheme consider a case where $I_n = \{i - 1, i + 1\}$ and function values are prescribed at another I_m nodes with node $i \in I_m$. The interpolation polynomial for this case can be obtained as

Table 1
Compact schemes for first derivative

Index i	I_n, I_m	Uniform grid $x_i = x_1 + h(i - 1)$	Non-uniform grid $x_i = x_1 + h(i - 1)^2$
1	$\{3,4\}$, $\{1,2\}$	$u_1' + 3u_2' = -\frac{17}{6h}u_1 + \frac{3}{2h}u_2 + \frac{3}{2h}u_3 - \frac{1}{6h}u_4$	$u_1' + \frac{3}{2}u_2' = -\frac{85}{36h}u_1$ $+\frac{111}{48h}u_2 + \frac{1}{20h}u_3 - \frac{1}{720h}u_4$
$2,3,\ldots,N-1$	$\{i-1, i+1\}$, $\{i\}$	$\frac{1}{4}u_{i-1}' + u_i' + \frac{1}{4}u_{i+1}' = \frac{3}{4h}(u_{i+1} - u_{i-1})$	$\frac{(2i-1)^2}{16(i-1)^2}u_{i-1}' + u_i' + \frac{(2i-3)^2}{16(i-1)^2}u_{i+1}' = \frac{4}{(2i-3)(2i-1)h}u_i$ $-\frac{(2i-1)^2}{8h(i-1)^2}\{\frac{1}{2i-3} + \frac{1}{4(i-1)}\}u_{i-1} + \frac{(2i-3)^2}{8h(i-1)^2}\{\frac{1}{2i-1} + \frac{1}{4(i-1)}\}u_{i+1}$
N	$\{N-2, N-3\}$,$\{N, N-1\}$	$u_N' + 3u_{N-1}' = \frac{17}{6h}u_N - \frac{3}{2h}u_{N-1} - \frac{3}{2h}u_{N-2} + \frac{1}{6h}u_{N-3}$	$u_N' + \frac{3(N-2)}{N-3}u_{N-1}' = \{\frac{2}{2N-3} + \frac{1}{2(2N-4)} + \frac{1}{3(2N-5)}\}\{\frac{1}{h}\}u_N$ $+\{\frac{1}{2N-5} + \frac{1}{2(2N-6)} - \frac{2}{2N-3}\}\{\frac{1}{h}\} \times \frac{3(N-2)}{N-3}u_{N-1}$ $-\frac{3(2N-3)^2}{4h(N-2)(2N-5)(2N-7)}u_{N-2}$ $+\frac{(2N-4)(2N-3)^2}{6h(2N-5)(2N-7)(2N-6)^2}u_{N-3}$

$$
\begin{aligned}
u(x) = \sum_{j\in I_m} & \left\{ \frac{(x-x_{i+1})(x-x_{i-1})}{(x_j-x_{i+1})(x_j-x_{i-1})} \right\} l_j^m(x) \left[1 + C_{1j}(x-x_j) + C_{2j}(x-x_j)^2\right] u_j \\
& + \left\{ \frac{(x-x_{i+1})\prod_m(x)}{(x_{i-1}-x_{i+1})\prod_m(x_{i-1})} \right\} \left[1 + A_1^-(x-x_{i-1}) + A_2^-(x-x_{i-1})^2\right] u_{i-1} \\
& + \left\{ \frac{(x-x_{i-1})\prod_m(x)}{(x_{i+1}-x_{i-1})\prod_m(x_{i+1})} \right\} \left[1 + A_1^+(x-x_{i+1}) + A_2^+(x-x_{i+1})^2\right] u_{i+1} \\
& + \left\{ \frac{(x-x_{i+1})\prod_m(x)}{(x_{i-1}-x_{i+1})\prod_m(x_{i-1})} \right\} \left[B_1^-(x-x_{i-1}) + B_2^-(x-x_{i-1})^2\right] u''_{i-1} \\
& + \left\{ \frac{(x-x_{i-1})\prod_m(x)}{(x_{i+1}-x_{i-1})\prod_m(x_{i+1})} \right\} \left[B_1^+(x-x_{i+1}) + B_2^+(x-x_{i+1})^2\right] u''_{i+1}
\end{aligned}
$$

Table 2
Coefficients for interpolation polynomial and second derivative tridiagonal scheme for interior grid points

D	$=$	$6 + 4(x_{i+1}-x_{i-1})\left\{\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})} - \frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}\right\} - 2(x_{i+1}-x_{i-1})^2 \frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}$
A_1^+D	$=$	$-4\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})} - 2\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})} + 2(x_{i+1}-x_{i-1})\left\{\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})} - \frac{\prod_m''(x_{i+1})}{\prod_m(x_{i+1})}\right\} + (x_{i+1}-x_{i-1})^2\frac{\prod_m''(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}$
A_2^+D	$=$	$4\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})} - \frac{\prod_m''(x_{i+1})}{\prod_m(x_{i+1})} - \frac{2}{(x_{i+1}-x_{i-1})}\left\{\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})} - \frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}\right\} + (x_{i+1}-x_{i-1})\frac{\prod_m''(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}$
A_1^-D	$=$	$-4\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})} - 2\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})} + 2(x_{i-1}-x_{i+1})\left\{\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})} - \frac{\prod_m''(x_{i-1})}{\prod_m(x_{i-1})}\right\} + (x_{i-1}-x_{i+1})^2\frac{\prod_m''(x_{i-1})}{\prod_m(x_{i-1})}\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}$
A_2^-D	$=$	$4\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})} - \frac{\prod_m''(x_{i-1})}{\prod_m(x_{i-1})} - \frac{2}{(x_{i-1}-x_{i+1})}\left\{\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})} - \frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\right\} + (x_{i-1}-x_{i+1})\frac{\prod_m''(x_{i-1})}{\prod_m(x_{i-1})}\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}$
B_1^+D	$=$	$-\left\{\frac{2}{(x_{i-1}-x_{i+1})} + \frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}\right\}(x_{i-1}-x_{i+1})^2$
B_2^+D	$=$	$1 - (x_{i+1}-x_{i-1})\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}$
B_1^-D	$=$	$-\left\{\frac{2}{(x_{i+1}-x_{i-1})} + \frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\right\}(x_{i+1}-x_{i-1})^2$
B_2^-D	$=$	$1 - (x_{i-1}-x_{i+1})\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}$
$C_{1j}D$	$=$	$\frac{(x_{i+1}+x_{i-1}-2x_j)}{(x_{i+1}-x_j)(x_j-x_{i-1})}\left\{10 + \frac{4(x_{i+1}-x_{i-1})^2}{(x_{i+1}-x_j)(x_{i-1}-x_j)}\right\} + 2(x_{i+1}-x_{i-1})\left\{\frac{x_{i+1}-x_j}{x_{i-1}-x_j} - \frac{x_{i-1}-x_j}{x_{i+1}-x_j}\right\}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}$ $+\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}\left\{4\frac{x_{i+1}-x_{i-1}}{x_{i-1}-x_j} + 4\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_j} - 2\frac{(x_{i+1}-x_{i-1})^2}{(x_{i+1}-x_j)^2}\right\} - \frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\left\{4\frac{x_{i+1}-x_{i-1}}{x_{i-1}-x_j} + 4\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_j} + 2\frac{(x_{i+1}-x_{i-1})^2}{(x_{i-1}-x_j)^2}\right\}$
$C_{2j}D$	$=$	$2\left\{\frac{1}{(x_{i+1}-x_j)^2} + \frac{1}{(x_{i-1}-x_j)^2} + \frac{1}{(x_{i-1}-x_j)(x_{i+1}-x_j)}\right\} - 2\frac{(x_{i+1}-x_{i-1})^2}{(x_{i+1}-x_j)(x_{i-1}-x_j)}\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}$ $-2\frac{\prod_m'(x_{i+1})}{\prod_m(x_{i+1})}\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_j}\left\{\frac{1}{x_{i-1}-x_j} + \frac{1}{x_{i+1}-x_j}\right\} - 2\frac{\prod_m'(x_{i-1})}{\prod_m(x_{i-1})}\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_j}\left\{\frac{1}{x_{i-1}-x_j} + \frac{1}{x_{i+1}-x_j}\right\}$
a_{i-1}	$=$	$\frac{2\prod_m'(x_i)\{B_1^-(2x_i-x_{i+1}-x_{i-1})+B_2^-(x_i-x_{i-1})(3x_i-2x_{i+1}-x_{i-1})\}+\prod_m''(x_i)(x_i-x_{i+1})(x_i-x_{i-1})\{B_1^-+(x_i-x_{i-1})B_2^-\}}{(x_{i+1}-x_{i-1})\prod_m(x_{i-1})}$
a_{i+1}	$=$	$\frac{2\prod_m'(x_i)\{B_1^+(2x_i-x_{i+1}-x_{i-1})+B_2^+(x_i-x_{i+1})(3x_i-2x_{i-1}-x_{i+1})\}+\prod_m''(x_i)(x_i-x_{i+1})(x_i-x_{i-1})\{B_1^++(x_i-x_{i+1})B_2^+\}}{(x_{i-1}-x_{i+1})\prod_m(x_{i+1})}$
b_{i-1}	$=$	$\frac{2\prod_m'(x_i)\{1+A_1^-(2x_i-x_{i+1}-x_{i-1})+A_2^-(x_i-x_{i-1})(3x_i-2x_{i+1}-x_{i-1})\}+\prod_m''(x_i)(x_i-x_{i+1})\{1+A_1^-(x_i-x_{i-1})+A_2^-(x_i-x_{i-1})^2\}}{(x_{i-1}-x_{i+1})\prod_m(x_{i-1})}$
b_{i+1}	$=$	$\frac{2\prod_m'(x_i)\{1+A_1^+(2x_i-x_{i+1}-x_{i-1})+A_2^+(x_i-x_{i+1})(3x_i-2x_{i-1}-x_{i+1})\}+\prod_m''(x_i)(x_i-x_{i-1})\{1+A_1^+(x_i-x_{i+1})+A_2^+(x_i-x_{i+1})^2\}}{(x_{i+1}-x_{i-1})\prod_m(x_{i+1})}$
b_i	$=$	$\left(2C_{2i} + 2C_{1i}\left\{\frac{2x_i-x_{i-1}-x_{i+1}}{(x_i-x_{i+1})(x_i-x_{i-1})} + l_i^{m'}(x_i)\right\} + \left\{\frac{2+2l_i^{m'}(x_i)(2x_i-x_{i+1}-x_{i-1})}{(x_i-x_{i+1})(x_i-x_{i-1})} + l_i^{m''}(x_i)\right\}\right)$
b_j	$=$	$\left[\left\{C_{1j}\left(1+\frac{x_i-x_j}{x_i-x_{i+1}}+\frac{x_i-x_j}{x_i-x_{i-1}}\right) + C_{2j}\left(2+\frac{x_i-x_j}{x_i-x_{i+1}}+\frac{x_i-x_j}{x_i-x_{i-1}}\right)(x_i-x_j) + \frac{1}{x_i-x_{i+1}} + \frac{1}{x_i-x_{i-1}}\right\} \times\right.$ $\left. 2l_j^{m'}(x_i)\frac{(x_i-x_{i+1})(x_i-x_{i+1})}{(x_j-x_{i+1})(x_j-x_{i+1})} + \left\{\frac{(x_i-x_{i+1})(x_i-x_{i-1})}{(x_j-x_{i+1})(x_j-x_{i-1})}\right\}\left\{1+C_{1j}(x_i-x_j)+C_{2j}(x_i-x_j)^2\right\} l_j^{m''}(x_i)\right]$

with the condition for existence of polynomial given by

$$3+2(x_{i+1}-x_{i-1})\left\{\frac{\prod'_m(x_{i+1})}{\prod_m(x_{i+1})}-\frac{\prod'_m(x_{i-1})}{\prod_m(x_{i-1})}\right\}-(x_{i+1}-x_{i-1})^2\frac{\prod'_m(x_{i+1})}{\prod_m(x_{i+1})}\frac{\prod'_m(x_{i-1})}{\prod_m(x_{i-1})}\neq 0.$$

The general form of the tridiagonal scheme, obtained by differentiating above polynomial twice and then evaluating the derivative at $x = x_i$, is

$$a_{i-1}u''_{i-1}+u''_i+a_{i+1}u''_{i+1}=b_{i-1}u_{i-1}+b_iu_i+b_{i+1}u_{i+1}+\sum_{j\in I_m\neq i}b_ju_j. \tag{15}$$

The coefficients of the interpolation polynomial A_1^-, A_2^-, A_1^+, A_2^+, B_1^-, B_2^-, B_1^+, B_2^+, C_{1j} and C_{2j} and the scheme a_{i-1}, a_{i+1}, b_{i-1}, b_i, b_{i+1} and b_j are given in Table 2. The order of the scheme if I_m has m points will be $m + 2$. For deriving boundary scheme corresponding to the tridiagonal interior scheme consider a case in which $I_n = 2$ and function values are prescribed at another I_m nodes with node $1 \in I_m$. The interpolation polynomial for this case is

$$\begin{aligned}u(x)={}&\sum_{j\in I_m}\left(\frac{x-x_2}{x_j-x_2}\right)l_j^m(x)\left\{1-\frac{(x-x_j)l_j^{m'}(x_2)}{l_j^m(x_2)+(x_2-x_j)l_j^{m'}(x_2)}\right\}u_j+\frac{\prod_m(x)}{\prod_m(x_2)}\left\{1-\frac{x-x_2}{2}\frac{\prod''_m(x_2)}{\prod'_m(x_2)}\right\}u_2\\&+\frac{x-x_2}{2}\frac{\prod_m(x)}{\prod'_m(x_2)}u''_2\end{aligned}$$

with the condition for existence of polynomial given by

$$\frac{\prod'_m(x_2)}{\prod_m(x_2)}=l_j^m(x_2)+(x_2-x_j)l_j^{m'}(x_2)\neq 0.$$

The form of boundary scheme then obtained by differentiating above polynomial twice and then evaluating the second derivative at $x = x_1$ is

$$\begin{aligned}&u''_1+a_2u''_2=b_1u_1+b_2u_2+\sum_{j\in I_m\neq 1}b_ju_j,\\&a_2=\left\{\frac{x_2-x_1}{2}\frac{\prod''_m(x_1)}{\prod'_m(x_2)}-\frac{\prod'_m(x_1)}{\prod'_m(x_2)}\right\},\\&b_1=l_1^{m''}(x_1)+\frac{2l_1^{m'}(x_1)}{x_1-x_2}\left\{\frac{l_1^m(x_2)+2(x_2-x_1)l_1^{m'}(x_2)}{l_1^m(x_2)+(x_2-x_1)l_1^{m'}(x_2)}\right\}+\frac{2l_1^{m'}(x_2)}{(x_2-x_1)l_1^m(x_2)+(x_2-x_1)^2l_1^{m'}(x_2)},\\&b_2=\left\{\frac{\prod''_m(x_1)}{\prod_m(x_2)}+\frac{x_2-x_1}{2}\frac{\prod''_m(x_1)\prod''_m(x_2)}{\prod_m(x_2)\prod'_m(x_2)}-\frac{\prod'_m(x_1)}{\prod'_m(x_2)}\frac{\prod''_m(x_2)}{\prod_m(x_2)}\right\},\\&b_j=l_j^{m''}(x_1)\frac{(x_1-x_2)l_j^m(x_2)-(x_2-x_1)^2l_j^{m'}(x_2)}{(x_j-x_2)l_j^m(x_2)-(x_2-x_j)^2l_j^{m'}(x_2)}+\frac{2l_j^{m'}(x_1)}{x_j-x_2}\frac{l_j^m(x_2)+2(x_2-x_1)l_j^{m'}(x_2)}{l_j^m(x_2)+(x_2-x_j)l_j^{m'}(x_2)}\end{aligned} \tag{16}$$

For example third-order accurate second derivative tridiagonal compact schemes for interior and boundary points for a uniform and a non-uniform grid with the distribution of nodes given by $x_i = x_1 + h(i - 1)$, $i = 1,2,\ldots,N$ and $x_i = x_1 + h(i - 1)^2$, $i = 1,2,\ldots,N$, respectively, are presented in Table 3. The choice of sets I_n and I_m needed to derive these schemes is the same as those for the first derivative schemes. Two additional examples of second derivative compact finite difference schemes on uniform and non-uniform grid are listed in Table B.2.

Table 3
Compact schemes for second derivative

Index i	Uniform Grid $x_i = x_1 + h(i-1)$	Non-uniform Grid $x_i = x_1 + h(i-1)^2$
1	$u_1'' + 44u_2'' = \frac{13}{h^2}u_1 - \frac{27}{h^2}u_2 + \frac{15}{h^2}u_3 - \frac{1}{h^2}u_4$	$u_1'' - \frac{196}{13}u_2'' = -\frac{119}{78h^2}u_1 + \frac{107}{52h^2}u_2 - \frac{71}{130h^2}u_3 + \frac{11}{780h^2}u_4$
$2, 3, \ldots, N-1$	$\frac{1}{10}u_{i-1}'' + u_i'' + \frac{1}{10}u_{i+1}'' = \frac{6}{5h^2}(u_{i-1} + u_{i-1}) - \frac{12}{5h^2}u_i$	$\frac{(2i-1)(4i^2-16i+11)}{4(i-1)(20i^2-40i+19)}u_{i-1}'' + u_i'' + \frac{(2i-3)(4i^2-5)}{4(i-1)(20i^2-40i+19)}u_{i+1}'' = \frac{3(2i-1)}{(i-1)(20i^2-40i+19)h^2}u_{i-1} - \frac{12}{(20i^2-40i+19)h^2}u_i + \frac{3(2i-3)}{(i-1)(20i^2-40i+19)h^2}u_{i+1}$
N	$u_N'' + 44u_{N-1}'' = \frac{13}{h^2}u_N - \frac{27}{h^2}u_{N-1} + \frac{15}{h^2}u_{N-2} - \frac{1}{h^2}u_{N-3}$	$u_N'' + \frac{12C(N-2)}{N-3}u_{N-1}'' = \{\frac{12N-26}{3(2N-3)(2N-4)(2N-5)} + \frac{(6N-17)C}{(2N-3)(2N-5)(2N-6)}\} \times\{\frac{1}{h^2}\}u_N - \{\frac{10N-23}{(2N-3)(2N-5)(2N-6)} + \frac{3C(4N-14)(2N-4)}{(2N-3)(2N-5)(2N-6)^2}\}\{\frac{1}{h^2}\}u_{N-1} + \{\frac{8N-18}{(2N-4)(2N-5)(2N-7)} + \frac{3C(2N-9)}{(2N-5)(2N-6)(2N-7)}\} \times\{\frac{1}{h^2}\}u_{N-2} - \{\frac{6N-11}{3(2N-5)(2N-6)(2N-7)} - \frac{2(2N-4)C}{(2N-5)(2N-6)^2(2N-7)}\}\{\frac{1}{h^2}\}u_{N-3}$

$$\text{where}\quad C = \left\{\frac{1}{2N-3} + \frac{1}{2(2N-4)} + \frac{1}{3(2N-5)}\right\} \Big/ \left\{\frac{1}{2N-5} + \frac{1}{2(2N-6)} - \frac{1}{2N-3}\right\}.$$

In general, interpolation polynomial for constructing pentadiagonal and compact schemes with longer stencil for the derivative can be obtained numerically using the technique discussed in [2]. This will involve inversion of an $n \times n$ matrix which will always be computationally less expensive than using method of undetermined coefficients which requires solution of $2n + m - 1$ simultaneous linear algebraic equations.

3. High-order non-uniform grid schemes with boundary closure

3.1. Grid-spacing

The main limiting factor in the application of high-order (up to 14th order) compact schemes in practical computations is the numerical instability of the high-order boundary closure schemes. The high-order finite difference schemes are based on use of high-order polynomial interpolation which are known to show oscillations near the boundary when a uniform grid is used. It has been shown in [4] that clustering of grid points near the boundary for the case of high-order finite difference schemes enables use of boundary closure schemes which are of the same order as the interior. Here we follow the same approach and control the grid spacing using the stretching function proposed by Kosloff and Tal-Ezer [16] for a spectral method,

$$x_i = \frac{\sin^{-1}(-\alpha\cos(\pi i/N))}{\sin^{-1}\alpha}, \quad i = 0, \ldots, N, \tag{17}$$

where the parameter α is used to change the stretching of the grid points from one limit of a Chebyshev grid at $\alpha \rightarrow 0$ and the other limit of a uniform grid at $\alpha = 1$. Hence, by controlling the value of α the clustering of points near the boundary can be controlled and an optimum value of α for which the scheme is stable can be found.

3.2. Asymptotic stability analysis

The asymptotic stability of the high-order compact schemes with boundary closures is analyzed by computing the eigenvalues of the matrices obtained by spatial discretization of the following wave equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{18}$$

in a fixed computational domain $(-1,1)$. The non-periodic boundary condition is specified at $x = -1$ to a fixed value $u(x = -1,t) = f(t)$. After a computational grid is assigned to the domain, the spatial derivatives at all grid points, including the interior and boundary points, are discretized by a compact finite difference scheme. Note that if the grid is not time varying, coefficients of the scheme can be calculated and stored once for all at the beginning of the computations. In addition, the boundary closures are derived using an increased stencil width compared to the interior schemes so that they have the same order of accuracy as the interior schemes. The derivatives at all grid points including interior and the boundary can be combined and written as

$$[P]\{u'\} = [Q]\{u\} \quad \text{or} \quad \{u'\} = [M]\{u\}. \tag{19}$$

Substituting the approximation (19) into the wave equation with the non-periodic boundary condition at $x = -1$ leads to

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = c\mathbf{M}\mathbf{u} + \mathbf{g}(\mathrm{t}). \tag{20}$$

The asymptotic stability condition for the semi-discrete equations is that all eigenvalues of matrix $\mathbf{M}$ contain no positive real parts. This is a necessary but not sufficient condition for the stability of long-time integration of the equation. Fig. 1(a) shows the eigenvalue spectrum for a 10th-order (7-5)pentadiagonal scheme on a uniform grid of 101 points. It can be observed that there are two eigenvalues in the unstable region of the spectrum. Thus a 10th-order pentadiagonal scheme with a 10th-order compact boundary closure will not be stable on a uniform grid. In order to stabilize the 10th-order scheme, a stretched grid given by (17) is used and it is found that the stability of the scheme improves as the grid becomes more and more stretched towards the boundary. Thus two unstable eigenvalues become less unstable for $\alpha = 0.9970$ and 0.9966 and are completely stable for $\alpha = 0.9965$ as shown in Fig. 1.

In addition, as the order increases the amount of stretching required for stability, which is also a function of the number of grid points N, increases. Fig. 2 shows the variation of grid stretching parameter α and corresponding $\Delta x_{\min}$ required to obtain stable boundary closure for various high-order compact schemes with the total number of points N. The $\Delta x_{\min}$ is normalized by grid spacing for a uniform grid with same number of grid points, i.e, $\Delta x_{\text{uniform}} = 2/N + 1$. The plots show that higher order schemes require smaller $\Delta x_{\min}$ for stability and as N increases, value of $\Delta x_{\min}/\Delta x_{\text{uniform}}$ approaches a constant value, which is consistent with the results of [4] for high-order finite difference schemes. It is also observed that the 12th-order (11-3)tridiagonal, (9-5)pentadiagonal and (7-7)septadiagonal schemes have nearly same stability requirements. The main advantage of this method compared to a spectral scheme is that minimum required grid spacing for obtaining stable schemes will be of $O(1/N)$ compared to $O(1/N^2)$ for spectral methods which leads to very severe timestep restrictions [4].

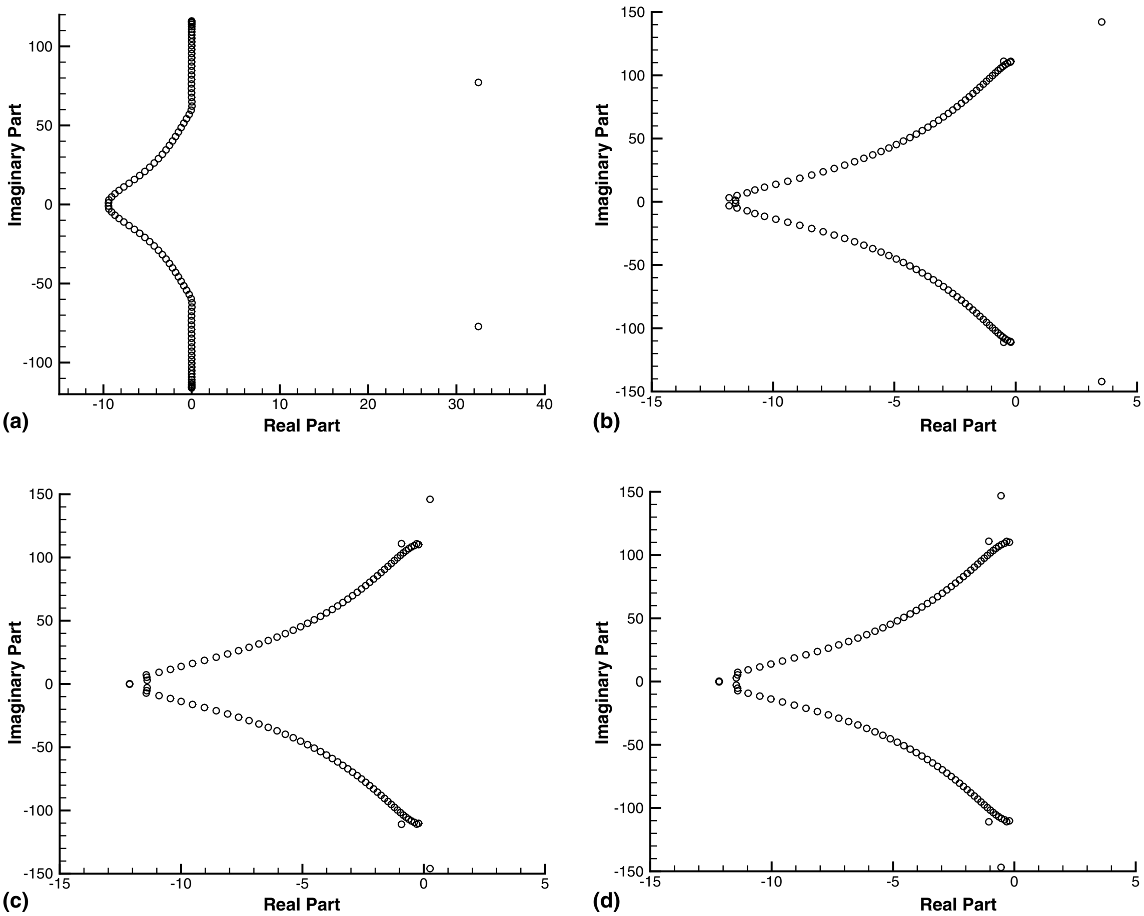

Fig. 1. Eigenvalue spectrum of the spatial discretization matrix for the 10th-order pentadiagonal scheme on (a) a uniform grid of 101 points, (b) a stretched grid of 101 points with $\alpha = 0.9970$, (c) a stretched grid of 101 points with $\alpha = 0.9966$, (d) a stretched grid of 101 points with $\alpha = 0.9965$.

3.3. Fourier analysis

The resolution ability of the schemes is studied by computing the dispersive and dissipative errors using a Fourier analysis. The trial function for this on a periodic domain is $u(x) = \mathrm{e}^{\mathrm{i}kx}$. The exact first and second derivatives of this trial function at nodes x_j are $\mathrm{i}k\mathrm{e}^{\mathrm{i}kx_j}$ and $-k^2\mathrm{e}^{\mathrm{i}kx_j}$. Application of the trial function to a first derivative compact scheme given by (14) leads to numerically computed first derivative of the form $\mathrm{i}k'\mathrm{e}^{\mathrm{i}kx_j}$, where

$$k' = -\mathrm{i}\frac{b_i + \sum_{j\in I_m\neq i} b_j \mathrm{e}^{\mathrm{i}k(x_j-x_i)} + \sum_{j\in I_n} c_j \mathrm{e}^{\mathrm{i}k(x_j-x_i)}}{1 + \sum_{j\in I_n} a_j \mathrm{e}^{\mathrm{i}k(x_j-x_i)}}. \tag{21}$$

The dispersive and dissipative errors are then given by the real part $\mathrm{Re}(k' - k)$ and the imaginary part Im-$(k' - k)$, respectively. The dispersive and dissipative errors will be different at various grid points for a non-uniform grid for a given compact scheme and here we present maximum dispersion and dissipation

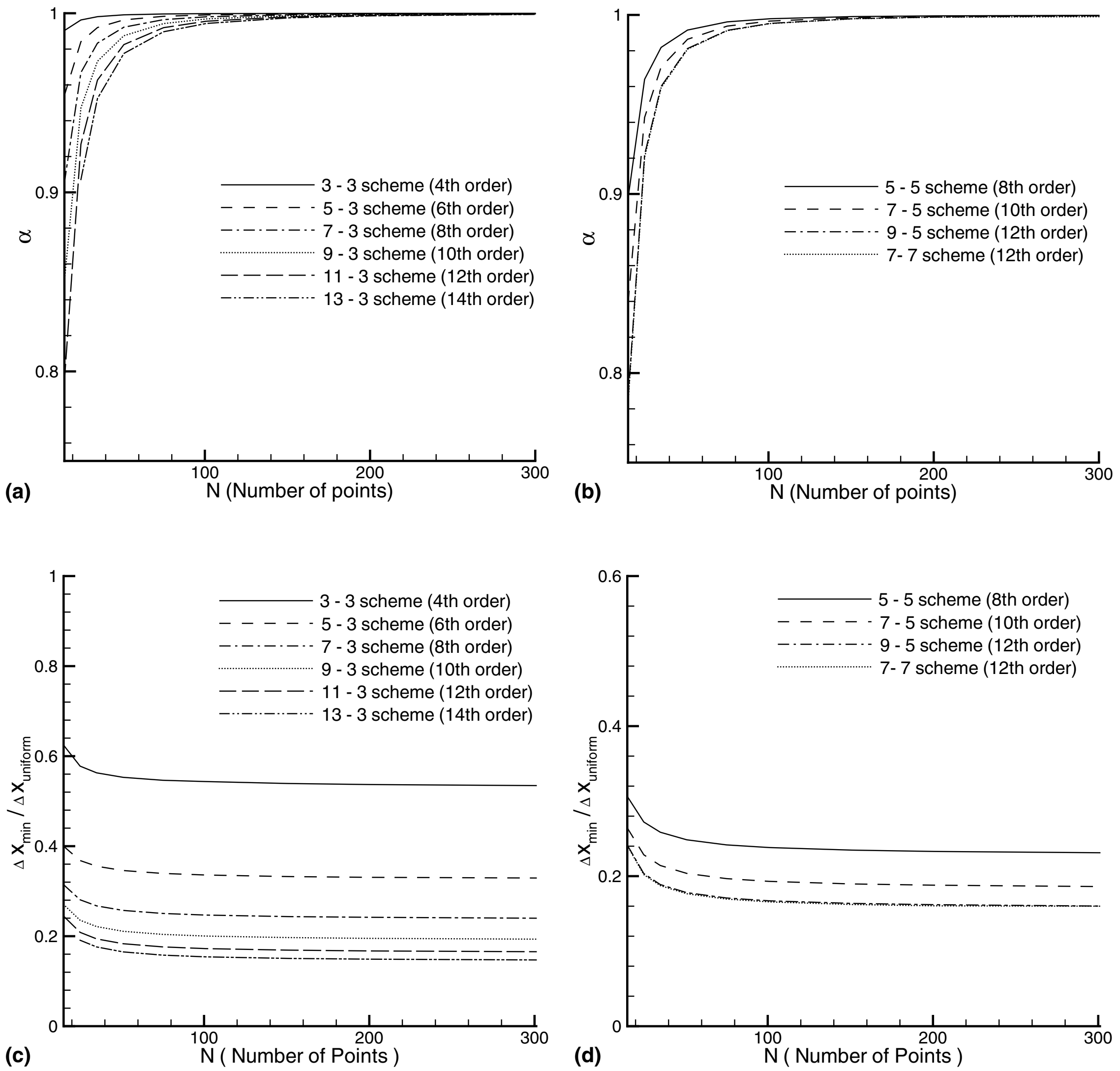

Fig. 2. (a and b) Variation of grid stretching parameter α required for stable boundary closure, with the total points used, for various compact schemes. (c and d) Variation of minimum Δx required for stable boundary closure with the total points used, for various compact schemes.

errors with x_j given by (17) for an alpha value of 0.9. The dispersion and dissipation plots of $k'h$ vs. kh for various first derivative compact schemes are shown in Fig. 3, where h is the largest grid spacing for the particular grid chosen. The plots show that resolution of the first derivative schemes improves as order is increased. As expected, the dispersion errors are reduced with increasing order of the schemes and also for the same order a pentadiagonal scheme has lesser dispersion error than a tridiagonal scheme and more dispersion error than a septadiagonal scheme. It is also found that the dissipation errors are non-zero, unlike the uniform grid schemes and increase with rising order of the scheme. In addition, for a given order a pentadiagonal scheme has more dissipation errors than the corresponding tridiagonal scheme and lesser dissipation errors than the septadiagonal scheme.

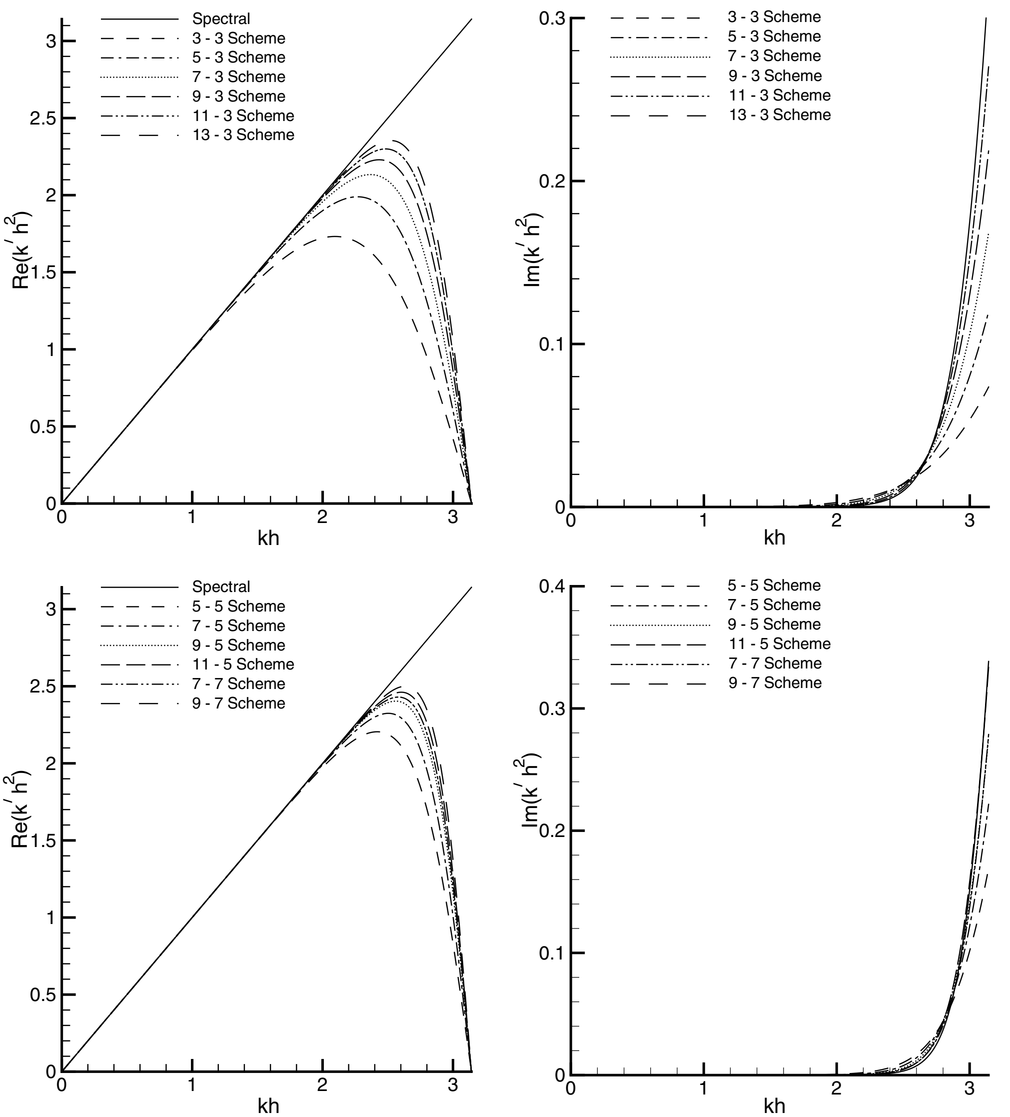

Fig. 3. Comparison of modified wavenumber for various non-uniform grid first derivative compact schemes.

Application of the trial function to a second derivative tridiagonal compact scheme given by (15) yields numerically computed second derivative of the form $-k''^2 e^{ikx_j}$, where

$$k''^2 = -\frac{b_i + b_{i-1}e^{ik(x_{i-1}-x_i)} + b_{i+1}e^{ik(x_{i+1}-x_i)} + \sum_{j\in I_m\neq i} b_j e^{ik(x_j-x_i)}}{1 + a_{i-1}e^{ik(x_{i-1}-x_i)} + a_{i+1}e^{ik(x_{i+1}-x_i)}}. \tag{22}$$

The dispersion and dissipation plots of k''^2h^2 vs kh for various tridiagonal second derivative compact schemes are shown in Fig. 4, where h is the largest grid spacing for the particular grid chosen.The plots show that the resolution of the second-derivative tridiagonal schemes improves with the increase in order along with a decrease in both dispersion and dissipation errors.

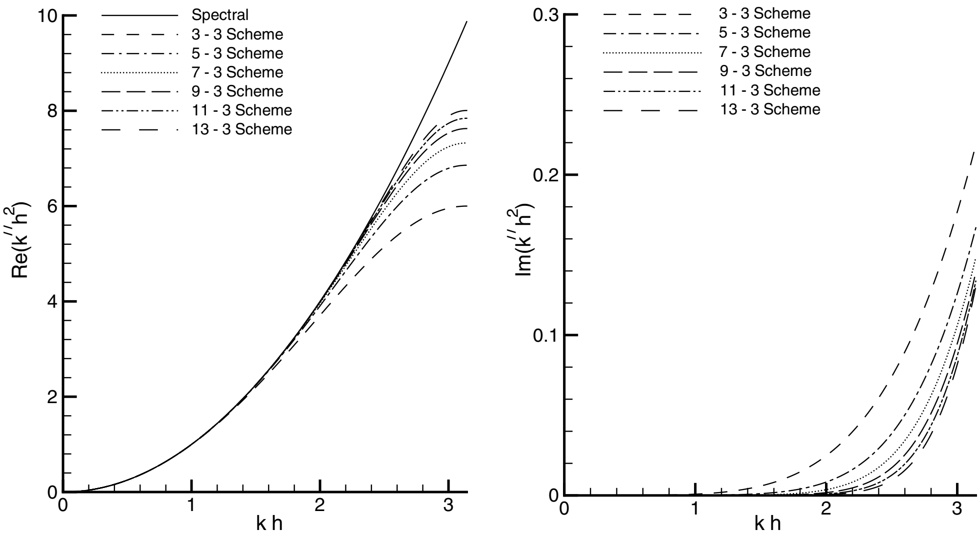

Fig. 4. Comparison of modified wavenumber for various non-uniform grid second derivative compact schemes.

4. Numerical results

The high-order compact schemes of upto 14th order developed in the earlier sections are used to solve one- and two-dimensional linear wave equation and a two-dimensional model convection–diffusion equation. For simplicity, only tridiagonal compact schemes are used for computations, however, we expect pentadiagonal and septadiagonal high-order schemes to follow similar trends. The results show that schemes are stable and offer very good accuracy.

4.1. One-dimensional wave equation computations

The one-dimensional wave equation given by (18) and with the non-periodic boundary condition at left boundary given by $u(-1,t) = \sin(\omega\pi t)$ is solved in a fixed computational domain (−1,1) with $c = 1$ and $\omega = 1$. Computations are performed using 4th- (3-3), 6th- (5-3), 8th- (7-3), 10th- (9-3), 12th- (11-3) and 14th- (13-3) order tridiagonal compact schemes. The time advancement is accomplished through a fourth-order Runge–Kutta scheme and the time step is chosen to be small enough so that the temporal errors are always smaller than the spatial errors. Fig. 5(a) shows a typical result on the comparison for the 10th-order tridiagonal scheme on a stretched grid with α equal to 0.9874 with the exact solution at time $t = 2.2$. The boundary closure is stable and there is excellent agreement between the numerical and exact solutions. On the other hand, computations using same scheme on a uniform grid are unstable as shown in Fig. 5(b). Figs. 6(b) and (c) show the growth of the average error, $\sqrt{\sum_{i=1}^{N}(u_i - u_{\text{exact}})^2/N}$, with time for a 10th-order tridiagonal compact scheme for three sets of grids having 25 points, 35 points and 51 points for uniform grid and a stretched grid with a value of grid stretching parameter α which makes the scheme just stable for a particular number of grid points. It can be observed that the error diverges exponentially for the case of uniform grid, whereas it remains bounded and stable for the stretched grid. The average error for various schemes for various grid sizes with a grid stretching parameter value equal to 0.8 is shown in Fig. 6(a). The value of stretching parameter α was chosen to be equal to 0.8 in all the cases so that stable schemes are obtained for all grids for all orders of accuracy. As expected, for a given number of grid points, the error decreases as the order of schemes is increased. In addition, for high N the

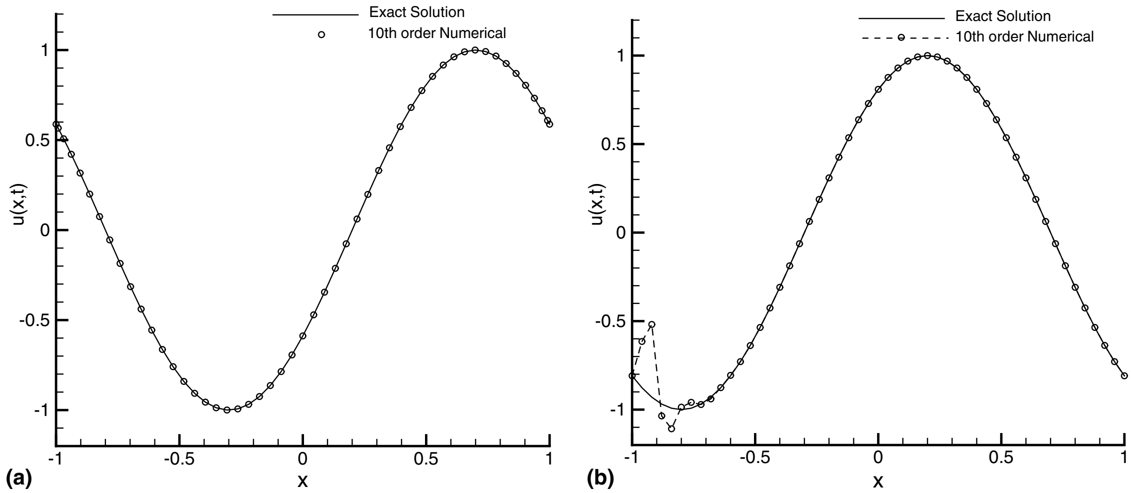

Fig. 5. Comparison of numerical solution with the exact solution using a 10th-order(9-3) tridiagonal compact scheme on (a) a stretched grid ($\alpha = 0.9874$) at $t = 2.2$ s and (b) a uniform grid at $t = 1.7$ s with 51 grid points.

error no longer decreases with increase in order due to the fact that numerical machine precision limit is reached.

4.2. Two-dimensional wave equation computations

In this section, we consider a model two-dimensional linear wave equation problem [10]

$$\begin{aligned} &\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0, \quad x\in[0,1],\ y\in[0,1],\ t\geqslant 0,\\ &u(0,y,t)=\sin\omega(y-2t), \quad u(x,0,t)=\sin\omega(x-2t),\\ &u(x,y,0)=\sin\omega(x+y). \end{aligned} \tag{23}$$

This problem has an analytical solution given by $u(x,y,t)=\sin\omega(x+y-2t)$. We solve this problem using 4th- (3-3), 6th- (5-3), 8th- (7-3) and 10th- (9-3) order tridiagonal compact schemes for 21×21, 31×31, 41×41 and 51×51 grid points with the non-uniform grid given by the following stretching function:

$$\begin{aligned} x_i&=\frac{1}{2}+\frac{\sin^{-1}(-\alpha\cos(\pi i/N))}{2\sin^{-1}\alpha}, \quad i=1,\ldots,N,\\ y_j&=\frac{1}{2}+\frac{\sin^{-1}(-\alpha\cos(\pi j/N))}{2\sin^{-1}\alpha}, \quad j=1,\ldots,N. \end{aligned} \tag{24}$$

The value of ω is set equal to 2π. The time advancement is accomplished through a fourth-order Runge–Kutta scheme and the time step is chosen to be small enough so that the temporal errors are always smaller than the spatial errors. The average error for all the schemes for various grid sizes with a grid stretching parameter value equal to 0.9 is shown in Fig. 7(a). As expected, for a given grid, error decreases as order of the scheme is increased. The value of stretching parameter parameter α was chosen to be equal to 0.9 in all cases so that the corresponding one-dimensional compact differentiation operator is stable for all the non-uniform grids. Note that it has been shown in [4] that the stability properties of the one-dimensional case are preserved for two-dimensional case if the differentiation operator matrix (represented by **M** in

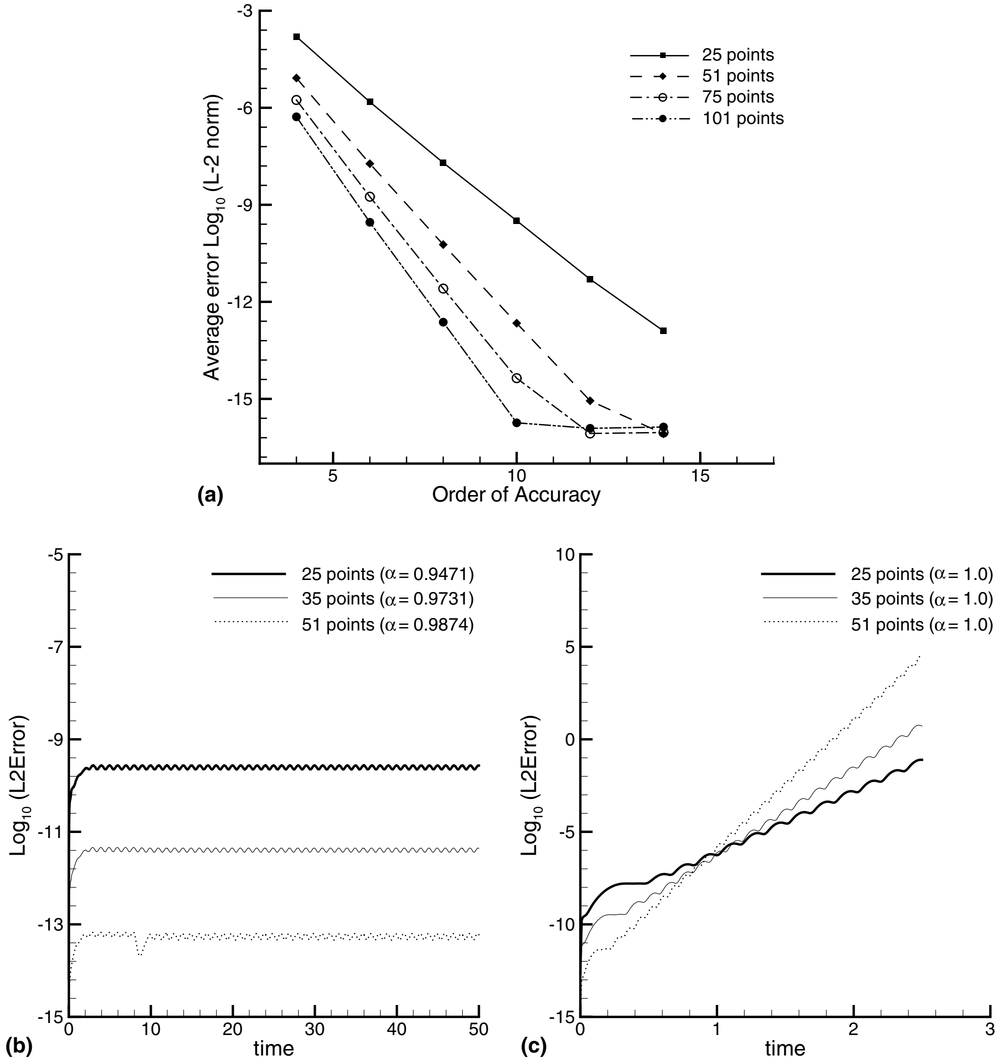

Fig. 6. (a) Average error for high-order tridiagonal scheme solutions of the one-dimensional wave equation. The L_2 error as a function of time using a 10th-order(9-3) tridiagonal compact scheme for solution of one-dimensional wave equation on (b) non-uniform grid and (c) uniform grid.

Section 3.2) has a complete set of eigenvectors. Figs. 7(b) and (c) show the growth of the average error, $\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}(u_{ij} - u_{\text{exact}})^2/N^2}$, with time for a 10th-order scheme for three sets of grids 21×21, 31×31 and 41×41 for both a uniform grid and a stretched grid with grid stretching parameter α equal to 0.9. It is observed that the error diverges exponentially for the case of uniform grid, whereas it remains bounded and stable for the stretched grid.

4.3. Two-dimensional convection–diffusion equation computations

The high-order non-uniform grid compact schemes are further tested by computing the linear decay of a two-dimensional convection–diffusion equation bounded by two parallel walls [34],

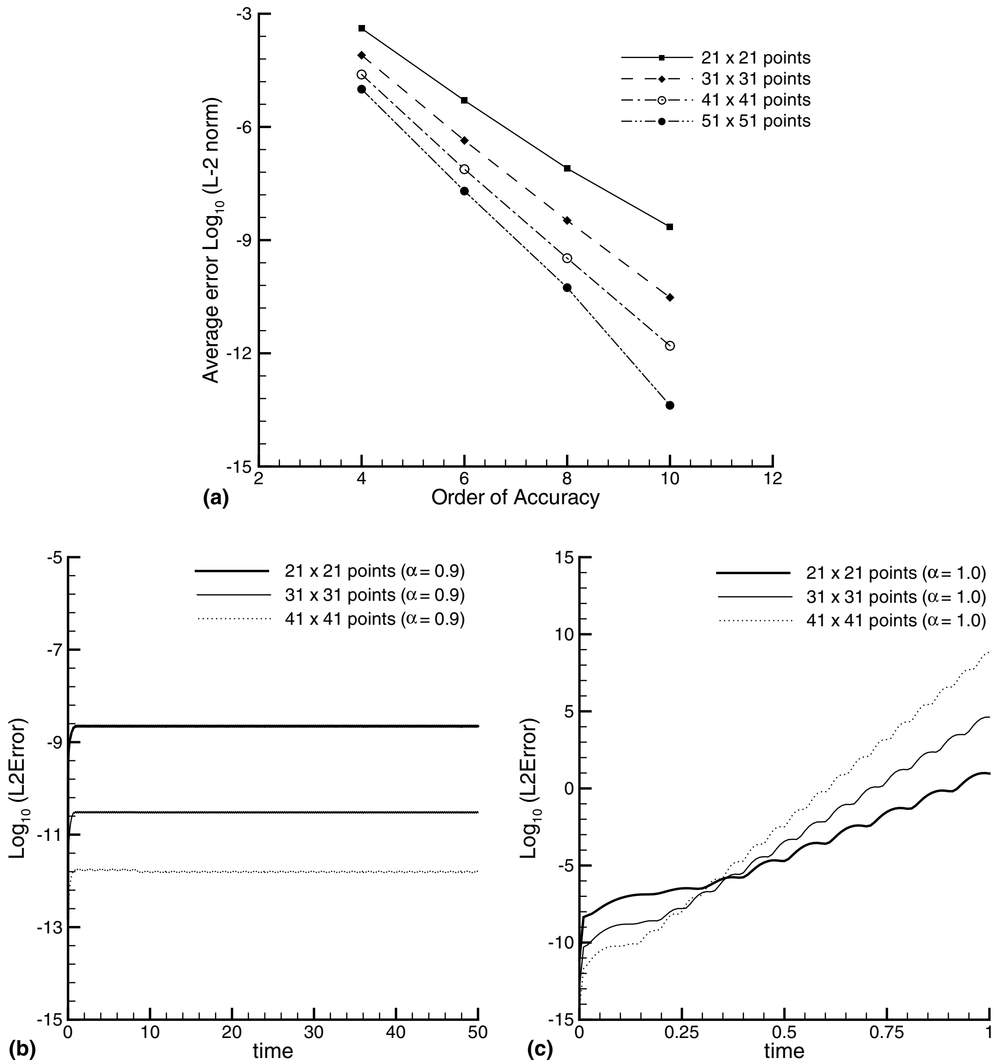

Fig. 7. (a) Average error for high-order tridiagonal scheme solutions of the two-dimensional wave equation. The L_2 error as a function of time using a 10th-order(9-3) tridiagonal compact scheme for solution of two-dimensional wave equation on (b) non-uniform grid and (c) uniform grid.

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{1}{R}\frac{\partial^2 u}{\partial y^2}. \tag{25}$$

The boundary conditions and the initial condition are

$$u(x,0)=u(x,1)=0,$$
$$u(0,y,t)=C\mathrm{e}^{Ry/2}\sin(n\pi y)sin(kx)\mathrm{e}^{-\alpha_n t},\quad \alpha_n=R(1+(2n\pi/R)^2)/4.$$

This model problem is used to test the accuracy of the high-order non-uniform second derivative tridiagonal schemes. The analytical solution for this problem is given by

$$u(x,y,t) = Ce^{Ry/2}\sin(n\pi y)\sin k(x-t)e^{-\alpha_n t}.$$

We solve this problem using 4th- (3-3), 6th- (5-3), 8th- (7-3) and 10th- (9-3) order tridiagonal compact schemes for first derivative and third- (3-3), fifth- (5-3), seventh- (7-3) and ninth- (9-3) order tridiagonal compact schemes for the second derivative, respectively. The values of parameters are $R = 10$, $k = 0.01$, $C = 1$, and $n = 3$. The computational domain, bounded by $(0,2\pi/k)\times(0,1)$, is discretized using 51×21, 51×31, 51×41 and 51×51 grid points with the non-uniform grid given by

$$\begin{aligned} x_i &= \frac{2\pi}{k}\left(\frac{1}{2}+\frac{\sin^{-1}(-\alpha\cos(\pi i/N))}{2\sin^{-1}\alpha}\right), \quad i=1,\ldots,N, \\ y_j &= \frac{1}{2}+\frac{\sin^{-1}(-\alpha\cos(\pi j/N))}{2\sin^{-1}\alpha}, \quad j=1,\ldots,M. \end{aligned} \tag{26}$$

The time advancement is accomplished through a fourth-order Runge–Kutta scheme and the time step is chosen to be small enough so that the temporal errors are always smaller than the spatial errors. For this

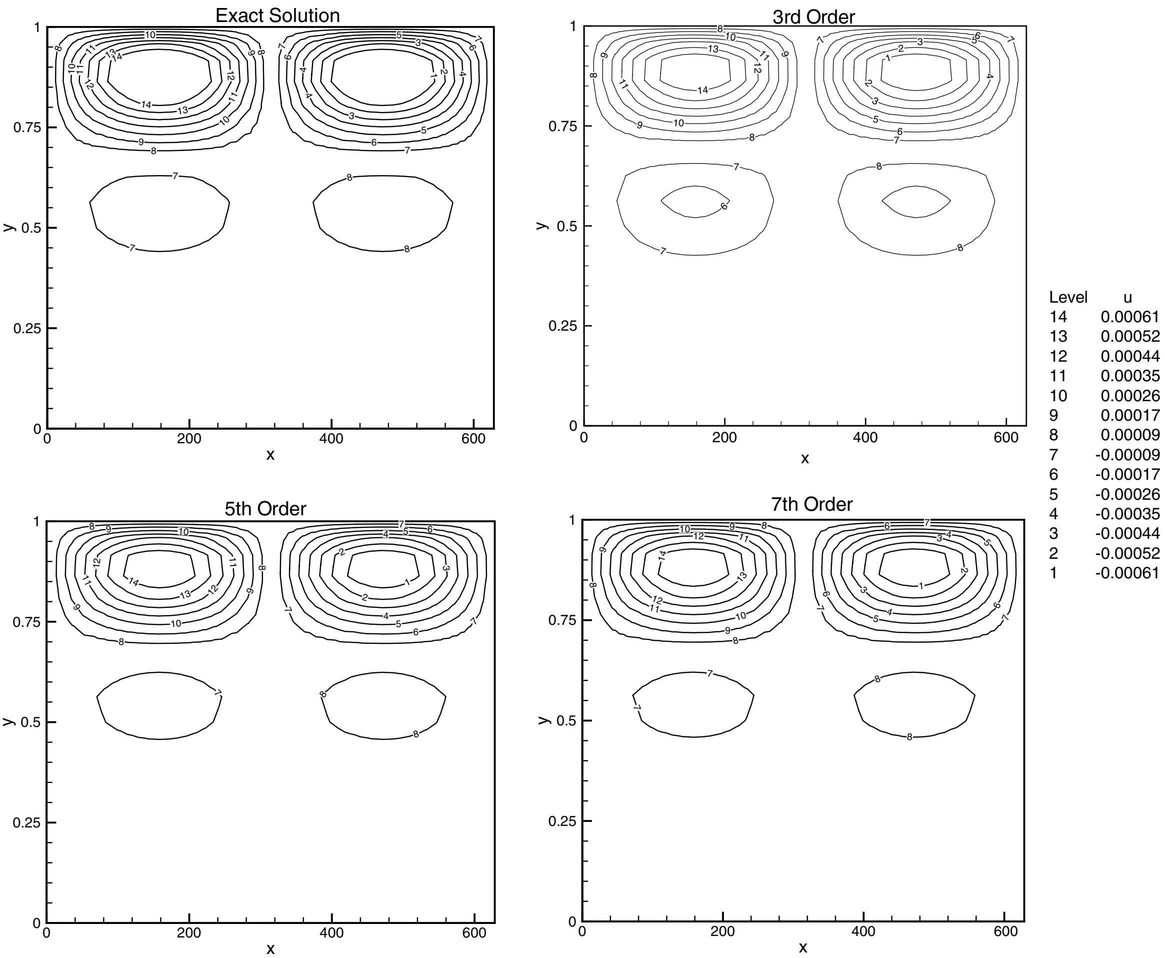

Fig. 8. The contours of instantaneous solution at t= 1 for 51×21 grid with $\alpha = 0.9$.

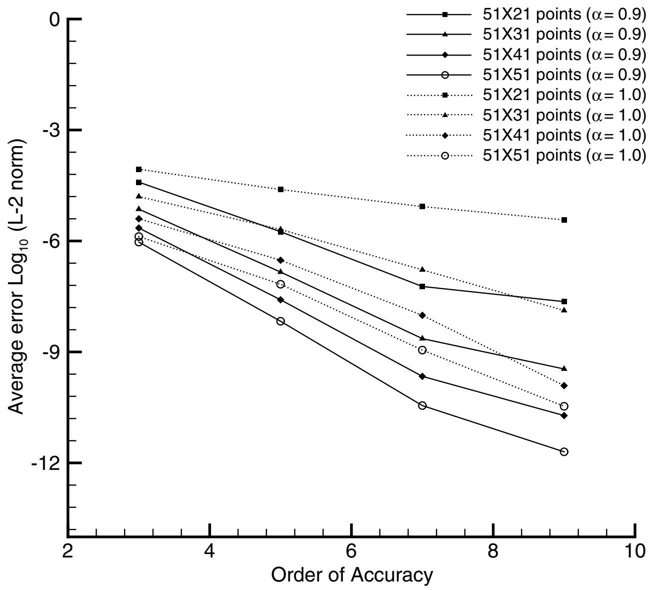

Fig. 9. Average error $\log_{10}$(L-2 norm) for high-order tridiagonal scheme solutions of the convection–diffusion equation on uniform ($\alpha = 1.0$) and non-uniform ($\alpha = 0.9$) grids.

particular problem the schemes are found to be stable on both uniform and non-uniform grids. Fig. 8 shows the comparison of contour plot of the instantaneous solution at time $t = 1$ for various high-order non-uniform grid compact schemes, with a grid stretching factor (α) value of 0.9, with the exact solution. The plot shows that fifth-order and seventh-order schemes are able to resolve the contour of the instantaneous solution better than the third-order scheme. Fig. 9 shows the plot of average error, $\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{M}(u_{ij} - u_{\text{exact}})^2/NM}$ vs. order for all the schemes for various grid sizes with grid stretching parameter values equal to 0.9 (non-uniform grid) and 1.0 (uniform grid) at time $t = 1$. As expected, for a given grid, error decreases as the order is increased both for uniform grid and non-uniform grid compact schemes.

5. Conclusions

In this paper simple polynomial interpolation is used to derive compact finite difference schemes over non-uniform grids with arbitrary grid spacing. For the case of first derivative an analytical relation is obtained for the scheme which is better than using Taylor expansion especially for time varying adaptive grids since there is no need to solve for the undetermined coefficients. The method can be easily extended to higher order compact schemes and the computational cost for determining coefficients of the interpolation polynomial will always be less than the cost of evaluating coefficients of the compact scheme using Taylor expansion. The high-order non-uniform grid schemes of up to 14th order along with the boundary closures of the same order as interior, derived using polynomial interpolation, are tested for solutions of wave equation in one and two dimensions and a model convection–diffusion equation. The results show that the schemes are stable and are able to produce highly accurate results provided enough grid points are clustered near the boundary.

Acknowledgement

The first author thanks Professor V.K. Dhir and Dr. M. Tatineni of the MAE Department at UCLA for their support and help during this research. The second author is sponsored by the Air Force Office of Scientific Research, USAF, under AFOSR Grant # FA 9550-04-1-0029, monitored by Dr. John Schmisseur.

Appendix A

In this section, we compare the computational cost of determining the coefficients of the compact schemes to the cost of obtaining the derivatives. A general compact finite difference formulation on a N point domain is of the form $[P]\{\hat{u}\} = [Q]\{u\}$, where function values(in vector form) $\{u\}$ are known and the unknown vector $\{\hat{u}\}$ can be $\{u'\}$ for first derivative, $\{u''\}$ for the second derivative and so on. In general matrices $[P]$ and $[Q]$ have the following form:

$$[P] = \begin{bmatrix} 1 & \cdots & a_{1(n+1)} & & & & & & 0 \\ a_{21} & 1 & \cdots & a_{2(n+2)} & & & & & \\ & & \cdots & \cdots & & & & & \\ & & & a_{(\frac{N}{2})(\frac{N}{2}-n)} & 1 & a_{(\frac{N}{2})(\frac{N}{2}+n)} & & & \\ & & & & \cdots & \cdots & & & \\ & & & & & a_{(N-1)(N-n-1)} & \cdots & 1 & a_{(N-1)(N)} \\ 0 & & & & & & a_{N(N-n)} & \cdots & 1 \end{bmatrix},$$

$$[Q] = \begin{bmatrix} b_{11} & \cdots & \cdots & b_{1(2m+3n+1)} & & & & & 0 \\ b_{21} & \cdots & b_{2(2m+3n)} & & & & & & \\ & & \cdots & \cdots & & & & & \\ & & & b_{(\frac{N}{2})(\frac{N}{2}-n-m)} & \cdots & b_{(\frac{N}{2})(\frac{N}{2}+n+m)} & & & \\ & & & & \cdots & \cdots & & & \\ & & & & & b_{(N-1)(N-2m-3n+1)} & & \cdots & b_{(N-1)N} \\ 0 & & & & b_{N(N-2m-3n)} & \cdots & & \cdots & b_{NN} \end{bmatrix}.$$

Thus matrices $[P]$ and $[Q]$ are both banded with bandwidths of $2n + 1$ and $2(n + m) + 1$, respectively. In addition, the width of the stencil at the boundary is increased so that both the interior and boundary closure schemes have the same order. Once coefficients of the matrices are known it is easy to find that the calculation of derivatives requires $N(n^2 + 5n + 2m + 2) + 2n(n + 1)$ multiplications and $N(n^2 + 4n + 2m) + 2n(n + 1)$ additions.

The cost of calculating coefficients of the matrix $[P]$ and matrix $[Q]$ for the first derivative is now obtained. Since the coefficients of the compact scheme involve product functions and their derivatives, we need to estimate the computational cost for calculation of these functions first. The formulae for calculation of the lagrange and product polynomials on a set of k points $\mathbf{I_k}$ and their first and second derivative along with the number of operations required are given in Table A.1.

Now let us consider the compact scheme given by (14) such that I_m has $p + 1$ points (including node i) and I_n has q points. Using Table A.1 the cost of calculation of coefficients for the compact scheme can be found out to be $9q^2 + 2p^2 + 9pq - p - 3q + 1$ multiplications and $10q^2 + 2p^2 + 10pq + p - 2q - 1$ additions. Then the cost of evaluating the coefficients of the matrices $[P]$ and $[Q]$ can be found out to be $N(36n^2 + 8m^2 + 36mn - 2m - 6n + 1) - n(n+1)(2m + \frac{2}{3}n - \frac{2}{3})$ multiplications and $N(40n^2 + 8m^2 + 40mn + 2m - 4n - 1) - 2n(n+1)(2m + \frac{2}{3}n - \frac{5}{3})$ additions. A comparison of the cost of calculating coefficients to the cost of calculating first derivative for some high-order first derivative compact schemes is presented in Table A.2.

Appendix B

Hermite interpolation can be used to obtain explicit expression for the combined compact schemes. For this let us consider the problem of finding interpolation polynomial for a case in which function values $u(x_i) = u_i$ have been specified on a set of points, $\mathbf{I_m}$, the values of both the function and its first derivative $u'(x_i) = u'_i$ have been specified at another set of points, $\mathbf{I_n}$, and the values of not only the function $u(x_i) = u_i$ and its first derivative but also the second derivative $u''(x_i) = u''_i$ have been specified on another set of

Table A.1
Computational cost of calculating polynomials

Polynomial	Mult.	Add.
$\prod_k(x_i) = \prod_{j\in I_k}(x_i - x_j)$	$k-1$	k
$\prod'_k(x_i) = \begin{cases} \prod_{j\in I_k, j\neq i}(x_i - x_j) & \text{if } x_i \in I_k \\ \prod_k(x_i)(\sum_{j\in I_k}\frac{1}{x_i - x_j}) & \text{if } x_i \notin I_k \end{cases}$	$k-2$ $2k$	$k-1$ $3k-1$
$\prod''_k(x_i) = \begin{cases} 2\prod_{j\in I_k, j\neq i}(x_i - x_j)(\sum_{j\in I_k, j\neq i}\frac{1}{x_i - x_j}) & \text{if } x_i \in I_k \\ \prod_k(x_i)((\sum_{j\in I_k}\frac{1}{x_i - x_j})^2 - \sum_{j\in I_k}\frac{1}{(x_i - x_j)^2}) & \text{if } x_i \notin I_k \end{cases}$	$2k-1$ $2k^2 - k$	$3k-4$ $3k^2 - 2k - 1$
$l^k_j(x_i) = \frac{\prod_{l\in I_k, l\neq j}(x_i - x_l)}{\prod_{l\in I_k, l\neq j}(x_j - x_l)}$	$2k-3$	$2k-2$
$l^{k'}_j(x_i) = \begin{cases} \frac{\prod_{l\in I_k, l\neq i,j}(x_i - x_l)}{\prod_{l\in I_k, l\neq j}(x_j - x_l)} & \text{if } x_i \in I_k \\ l^k_j(x_i)(\sum_{l\in I_k, l\neq j}\frac{1}{x_i - x_l}) & \text{if } x_i \notin I_k \end{cases}$	$2k-4$ $3k-3$	$2k-3$ $4k-5$
$l^{k''}_j(x_i) = \begin{cases} 2\frac{\prod_{l\in I_k, l\neq i,j}(x_i - x_l)}{\prod_{l\in I_k, l\neq j}(x_j - x_l)}(\sum_{l\in I_k, l\neq i,j}\frac{1}{x_i - x_l}) & \text{if } x_i \in I_k \\ l^k_j(x_i)((\sum_{l\in I_k \neq j}\frac{1}{x_i - x_l})^2 - \sum_{l\in I_k \neq j}\frac{1}{(x_i - x_l)^2}) & \text{if } x_i \notin I_k \end{cases}$	$3k-4$ $2k^2 - 4k + 2$	$4k-8$ $3k^2 - 7k + 3$

Table A.2
Computational cost comparison for calculation of coefficients and calculation of first derivative on a N point one-dimensional domain

Scheme	m	n	Cost for coefficients		Cost for derivative	
			Mult.	Add.	Mult.	Add.
4th-order Tridiagonal	0	1	$31N$	$35N + 4$	$8N + 4$	$5N + 4$
6th-order Tridiagonal	1	1	$73N - 4$	$85N - 4$	$10N + 4$	$7N + 4$
8th-order Tridiagonal	2	1	$131N - 8$	$151N - 12$	$12N + 4$	$9N + 4$
10th-order Tridiagonal	3	1	$205N - 12$	$233N - 20$	$14N + 4$	$11N + 4$
8th-order Pentadiagonal	0	2	$133N - 4$	$151N + 4$	$16N + 12$	$12N + 12$
10th-order Pentadiagonal	1	2	$211N - 16$	$241N - 20$	$18N + 4$	$14N + 12$

points, $\mathbf{I_k}$. The scheme for the second derivative can be derived by choosing $\mathbf{I_n} = \{\phi\}$ and then taking a double derivative of the interpolating polynomial at the point $x = x_i$

$$u_i'' + b_i u_i' + \sum_{j \in I_k} \left(a_j u_j'' + b_j u_j' \right) = c_i u_i + \sum_{j \in I_k} c_j u_j + \sum_{j \in I_m} \hat{c}_j u_j,$$

$$a_j = -\frac{\prod_m(x_i)}{\prod_m(x_j)} \left\{ l_j^k(x_i) \right\}^3,$$

$$b_i = -2\left\{ \frac{\prod_m'(x_i)}{\prod_m(x_i)} + 3\frac{\prod_k'(x_i)}{\prod_k(x_i)} \right\},$$

$$b_j = 2\frac{\prod_m(x_i)}{\prod_m(x_j)} \left\{ l_j^k(x_i) \right\}^3 \left\{ \frac{3}{x_j - x_i} + \frac{\prod_m'(x_j)}{\prod_m(x_j)} + 3 l_j^{k'}(x_j) \right\},$$

Table B.1
Compact schemes for first derivative on uniform and non-uniform stencils

Non-uniform grid	Uniform grid
$\left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^2 u'_{i-1} + u'_i + \left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^2 u'_{i+1} = 2\left\{\frac{1}{x_i-x_{i-1}} + \frac{1}{x_i-x_{i+1}}\right\} u_i +$ $2\left\{\frac{(x_i-x_{i+1})}{(x_{i-1}-x_{i+1})}\right\}^2 \left\{\frac{1}{x_{i-1}-x_i} + \frac{1}{x_{i-1}-x_{i+1}}\right\} u_{i-1} + 2\left\{\frac{(x_i-xi-1)}{(x_{i+1}-x_{i-1})}\right\}^2 \left\{\frac{1}{x_{i+1}-x_i} + \frac{1}{x_{i+1}-x_{i-1}}\right\} u_{i+1}$	$u'_{i+1} + 4u'_i + u'_{i-1}$ $= \frac{3}{h}(u_{i+1} - u_{i-1})$
$\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+1})^2}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+1})^2}\right\} u'_{i-1} + u'_i + \left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i-1})^2}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})(x_{i+1}-x_{i-1})^2}\right\} u'_{i+1} =$ $\left\{\frac{(x_i-x_{i+1})^2(x_i-x_{i-1})^2(x_i-x_{i+2})}{(x_{i-2}-x_{i+1})^2(x_{i-2}-x_{i-1})^2(x_{i-2}-x_i)(x_{i-2}-x_{i+2})}\right\} u_{i-2} + \left\{\frac{(x_i-x_{i+1})^2(x_i-x_{i-1})^2(x_i-x_{i-2})}{(x_{i+2}-x_{i+1})^2(x_{i+2}-x_{i-1})^2(x_{i+2}-x_i)(x_{i+2}-x_{i-2})}\right\} u_{i+2}$ $\left\{\frac{(x_i-x_{i+1})^2(x_i-x_{i-2})(x_i-x_{i+2})}{(x_{i-1}-x_{i+1})^2(x_{i-1}-x_{i-2})(x_{i-1}-x_i)(x_{i-1}-x_{i+2})}\right\} \left[2 + \left\{\frac{2(x_{i-1}-x_i)}{(x_{i-1}-x_{i+1})} + \frac{(x_{i-1}-x_i)}{(x_{i-1}-x_{i-2})} + \frac{(x_{i-1}-x_i)}{(x_{i-1}-x_{i+2})}\right\}\right] u_{i-1} +$ $\left\{\frac{(x_i-x_{i-1})^2(x_i-x_{i-2})(x_i-x_{i+2})}{(x_{i+1}-x_{i-1})^2(x_{i+1}-x_{i-2})(x_{i+1}-x_i)(x_{i+1}-x_{i+2})}\right\} \left[2 + \left\{\frac{2(x_{i+1}-x_i)}{(x_{i+1}-x_{i-1})} + \frac{(x_{i+1}-x_i)}{(x_{i+1}-x_{i-2})} + \frac{(x_{i+1}-x_i)}{(x_{i+1}-x_{i+2})}\right\}\right] u_{i+1} +$ $\left\{\frac{1}{x_i-x_{i+2}} + \frac{1}{x_i-x_{i-2}} + \frac{2}{x_i-x_{i+1}} + \frac{2}{x_i-x_{i-1}}\right\} u_i$	$\frac{1}{3}u'_{i-1} + u'_i + \frac{1}{3}u'_{i+1}$ $= \frac{7}{9h}(u_{i+1} - u_{i-1})$ $+ \frac{1}{36h}(u_{i+2} - u_{i-2})$
$\left\{\frac{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i-1})}{(x_{i-2}-x_{i+1})(x_{i-2}-x_{i+2})(x_{i-2}-x_{i-1})}\right\}^2 u'_{i-2} + \left\{\frac{(x_i-x_{i-1})(x_i-x_{i-2})(x_i-x_{i+1})}{(x_{i+2}-x_{i-1})(x_{i+2}-x_{i+1})(x_{i+2}-x_{i-2})}\right\}^2 u'_{i+2} +$ $\left\{\frac{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i-1}-x_{i+1})(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})}\right\}^2 u'_{i-1} + \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i+1}-x_{i-1})(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}\right\}^2 u'_{i+1} + u'_i =$ $\left\{\frac{2}{x_{i-2}-x_i}\right\} \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+1})(x_i-x_{i+2})}{(x_{i-2}-x_{i-1})(x_{i-2}-x_{i+1})(x_{i-2}-x_{i+2})}\right\}^2 \left\{1 + \frac{x_{i-2}-x_i}{x_{i-2}-x_{i-1}} + \frac{x_{i-2}-x_i}{x_{i-2}-x_{i+1}} + \frac{x_{i-2}-x_i}{x_{i-2}-x_{i+2}}\right\} u_{i-2} +$ $\left\{\frac{2}{x_{i+2}-x_i}\right\} \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+1})(x_i-x_{i-2})}{(x_{i+2}-x_{i-1})(x_{i+2}-x_{i+1})(x_{i+2}-x_{i-2})}\right\}^2 \left\{1 + \frac{x_{i+2}-x_i}{x_{i+2}-x_{i-1}} + \frac{x_{i+2}-x_i}{x_{i+2}-x_{i+1}} + \frac{x_{i+2}-x_i}{x_{i+2}-x_{i-2}}\right\} u_{i+2} +$ $\left\{\frac{2}{x_{i-1}-x_i}\right\} \left\{\frac{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i-1}-x_{i+1})(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})}\right\}^2 \left\{1 + \frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}} + \frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}} + \frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}\right\} u_{i-1} +$ $\left\{\frac{2}{x_{i+1}-x_i}\right\} \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i+1}-x_{i-1})(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}\right\}^2 \left\{1 + \frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}} + \frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}} + \frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}\right\} u_{i+1} +$ $2\left\{\frac{1}{x_i-x_{i-1}} + \frac{1}{x_i-x_{i+1}} + \frac{1}{x_i-x_{i-2}} + \frac{1}{x_i-x_{i+2}}\right\} u_i$	$\frac{4}{9}u'_{i+1} + u'_i + \frac{4}{9}u'_{i-1}$ $\frac{1}{36}u'_{i+2} + \frac{1}{36}u'_{i-2} =$ $\frac{25}{216h}(u_{i+2} - u_{i-2}) +$ $\frac{20}{27h}(u_{i+1} - u_{i-1})$
$\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+3})(x_i-x_{i-3})(x_i-x_{i+1})^2}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+3})(x_{i-1}-x_{i-3})(x_{i-1}-x_{i+1})^2}\right\} u'_{i-1} + \left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+3})(x_i-x_{i-3})(x_i-x_{i-1})^2}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})(x_{i+1}-x_{i+3})(x_{i+1}-x_{i-3})(x_{i+1}-x_{i-1})^2}\right\} u'_{i+1}$ $+u'_i = \left\{\frac{2}{x_i-x_{i-1}} + \frac{2}{x_i-x_{i+1}} + \frac{1}{x_i-x_{i-2}} + \frac{1}{x_i-x_{i+2}} + \frac{1}{x_i-x_{i-3}} + \frac{1}{x_i-x_{i+3}}\right\} u_i +$ $\left\{\frac{1}{x_{i-3}-x_i}\right\} \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+1})}{(x_{i-3}-x_{i-1})(x_{i-3}-x_{i+1})}\right\}^2 \left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+3})}{(x_{i-3}-x_{i+2})(x_{i-3}-x_{i-2})(x_{i-3}-x_{i+3})}\right\} u_{i-3} + \left\{\frac{1}{x_{i+3}-x_i}\right\} \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+1})}{(x_{i+3}-x_{i-1})(x_{i+3}-x_{i+1})}\right\}^2 \times$ $\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i-3})}{(x_{i+3}-x_{i+2})(x_{i+3}-x_{i-2})(x_{i+3}-x_{i-3})}\right\} u_{i+3} + \left\{\frac{1}{x_{i-2}-x_i}\right\} \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+1})}{(x_{i-2}-x_{i-1})(x_{i-2}-x_{i+1})}\right\}^2 \left\{\frac{(x_i-x_{i+2})(x_i-x_{i+3})(x_i-x_{i-3})}{(x_{i-2}-x_{i+2})(x_{i-2}-x_{i+3})(x_{i-2}-x_{i-3})}\right\} u_{i-2} +$ $\left\{\frac{1}{x_{i+2}-x_i}\right\} \left\{\frac{(x_i-x_{i-1})(x_i-x_{i+1})}{(x_{i+2}-x_{i-1})(x_{i+2}-x_{i+1})}\right\}^2 \left\{\frac{(x_i-x_{i-2})(x_i-x_{i+3})(x_i-x_{i-3})}{(x_{i+2}-x_{i-2})(x_{i+2}-x_{i+3})(x_{i+2}-x_{i-3})}\right\} u_{i+2} + \left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+3})(x_i-x_{i-3})}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+3})(x_{i-1}-x_{i-3})}\right\} \times$ $\left\{\frac{1}{x_{i-1}-x_i}\right\} \left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^2 \left\{2 + \frac{2(x_{i-1}-x_i)}{x_{i-1}-x_{i+1}} + \frac{x_{i-1}-x_i}{x_{i-1}-x_{i-3}} + \frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}} + \frac{x_{i-1}-x_i}{x_{i-1}-x_{i+3}} + \frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}\right\} u_{i-1} + \left\{\frac{1}{x_{i+1}-x_i}\right\} \left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^2 \times$ $\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+3})(x_i-x_{i-3})}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})(x_{i+1}-x_{i+3})(x_{i+1}-x_{i-3})}\right\} \left\{2 + \frac{2(x_{i+1}-x_i)}{x_{i+1}-x_{i-1}} + \frac{x_{i+1}-x_i}{x_{i+1}-x_{i-3}} + \frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}} + \frac{x_{i+1}-x_i}{x_{i+1}-x_{i+3}} + \frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}\right\} u_{i+1}$	$\frac{3}{8}u'_{i+1} + u'_i + \frac{3}{8}u'_{i-1}$ $= \frac{25}{32h}(u_{i+1} - u_{i-1})$ $+ \frac{1}{20h}(u_{i+2} - u_{i-2})$ $- \frac{1}{480h}(u_{i+3} - u_{i-3})$

$$c_i = \left\{ \frac{\prod_m''(x_i)}{\prod_m(x_i)} + 3\frac{\prod_k''(x_i)}{\prod_k(x_i)} - 2\left(\frac{\prod_m'(x_i)}{\prod_m(x_i)}\right)^2 - 12\left(\frac{\prod_k'(x_i)}{\prod_k(x_i)}\right)^2 - 6\frac{\prod_m'(x_i)}{\prod_m(x_i)}\,\frac{\prod_k'(x_i)}{\prod_k(x_i)}\right\},$$

$$c_j = \frac{2\left\{l_j^k(x_i)\right\}^3}{(x_i-x_j)^2}\left\{\frac{\prod_m(x_i)}{\prod_m(x_j)}\right\}\left\{6+3(x_j-x_i)\left(\frac{\prod_m'(x_j)}{\prod_m(x_j)}+3l_j^{k'}(x_j)\right)+(x_j-x_i)^2 \times\left[\left(\frac{\prod_m'(x_j)}{\prod_m(x_j)}\right)^2+6\left(l_j^{k'}(x_j)\right)^2+3\frac{\prod_m'(x_j)}{\prod_m(x_j)}l_j^{k'}(x_j)-\frac{\prod_m''(x_j)}{2\prod_m(x_j)}-\frac{3}{2}l_j^{k''}(x_j)\right]\right\},$$

$$\hat{c}_j = \frac{2l_j^m(x_i)}{(x_i-x_j)^2}\left\{\frac{\prod_k(x_i)}{\prod_k(x_j)}\right\}^3.$$

Table B.2
Compact schemes for second derivative on uniform and non-uniform stencils

Non-uniform grid	Uniform grid
$\left\{\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}\right\}\left\{\frac{(x_i-x_{i-1})^2+(x_{i+1}-x_i)(x_i-x_{i-1})-(x_{i+1}-x_i)^2}{(x_{i+1}-x_i)(x_i-x_{i-1})+(x_{i+1}-x_{i-1})^2}\right\}u''_{i-1}+\left\{\frac{(x_{i+1}-x_i)^2+(x_{i+1}-x_i)(x_i-x_{i-1})-(x_i-x_{i-1})^2}{(x_{i+1}-x_i)(x_i-x_{i-1})+(x_{i+1}-x_{i-1})^2}\right\}\times$ $\left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}u''_{i+1}+u''_i=\left\{\frac{12}{D}\right\}\left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}u_{i+1}-u_i+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}u_{i-1}\right\}$ where $D=(x_{i+1}-x_i)(x_i-x_{i-1})+(x_{i+1}-x_{i-1})^2$ and scheme exists if $D\neq 0$	$u''_{i-1}+10u''_i+u''_{i+1}=$ $\frac{12}{h^2}(u_{i-1}-2u_i+u_{i+1})$
$\{3-\frac{3(x_{i+1}-x_i)^2}{(x_{i+1}-x_{i-1})(x_i-x_{i-1})}+\left(\frac{3x_i-2x_{i-1}-x_{i+1}}{x_i-x_{i-1}}\right)\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}\right)\}\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})}\right\}\left\{\frac{2}{D}\right\}u''_{i-1}+$ $\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}\right\}\{\frac{3(x_i-x_{i-1})^2}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}-3+\left(\frac{3x_i-x_{i-1}-2x_{i+1}}{x_{i+1}-x_i}\right)\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}\right)\}\left\{\frac{2}{D}\right\}u''_{i+1}+$ $u''_i=[36+6\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}\right)\left(\frac{4x_{i+1}-3x_{i-1}-x_i}{x_{i+1}-x_{i-1}}\right)+6\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}\right)\left(\frac{3x_{i+1}-4x_{i-1}+x_i}{x_{i+1}-x_{i-1}}\right)+$ $12\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}\right)\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}\right)+\left(\frac{2(3x_{i+1}-x_{i+2}-x_{i-2}-x_i)}{(x_{i+1}-x_{i-2})(x_{i+1}-x_{i+2})}\right)\{3-3\frac{(x_i-x_{i-1})^2}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})}+$ $\left(2-\frac{x_i-x_{i-1}}{x_{i+1}-x_i}\right)\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}\right)\}]\frac{2(x_i-x_{i+2})(x_i-x_{i-2})}{D(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}u_{i+1}+[36+6\left(\frac{4x_{i+1}-3x_{i-1}-x_i}{x_{i+1}-x_{i-1}}\right)\times$ $\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}\right)+6\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}\right)\left(\frac{3x_{i+1}-4x_{i-1}+x_i}{x_{i+1}-x_{i-1}}\right)+12\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}\right)\times$ $\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}\right)+\{3-3\frac{(x_{i+1}-x_i)^2}{(x_i-x_{i-1})(x_{i+1}-x_{i-1})}+\left(2-\frac{x_{i+1}-x_i}{x_i-x_{i-1}}\right)\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}\right)\}\times$ $\left(\frac{2(3x_{i-1}-x_{i+2}-x_{i-2}-x_i)}{(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+2})}\right)]\frac{2(x_i-x_{i+2})(x_i-x_{i-2})}{D(x_{i-1}-x_i)(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})}u_{i-1}+\left\{\frac{2}{D}\right\}\left\{\frac{(x_i-x_{i+1})(x_i-x_{i-1})}{(x_{i-2}-x_{i+1})^2(x_{i-2}-x_{i-1})^2}\right\}[2\left\{\frac{x_{i-1}-x_{i+1}}{x_{i-2}-x_{i+2}}\right\}\times$ $\left\{\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_{i-1}}{x_{i-1}-x_{i-2}}\right\}\left\{\frac{x_{i+2}-x_i}{x_i-x_{i+1}}+\frac{x_{i+2}-x_i}{x_i-x_{i-1}}+\frac{2x_i-x_{i+1}-x_{i-1}}{x_{i-2}-x_i}\right\}+ab\left\{\frac{4(2x_i-x_{i+1}-x_{i-1})(x_i-x_{i+2})+2(x_{i+1}-x_i)(x_{i-1}-x_i)}{(x_{i-2}-x_{i+2})(x_{i-2}-x_i)}\right\}$ $+2a\{\left(\frac{x_{i-2}-x_{i-1}}{x_{i+1}-x_{i-2}}\right)\left\{\frac{x_i-x_{i-1}}{x_{i-2}-x_{i+2}}+\frac{x_i-x_{i+2}}{x_{i-2}-x_{i+2}}\left(3+\frac{x_i-x_{i-1}}{x_i-x_{i+1}}-\frac{x_{i-2}-x_{i-1}}{x_{i-2}-x_i}\right)\right\}+\left\{\frac{x_i-x_{i-1}}{x_{i-2}-x_i}+\frac{x_i-x_{i+2}}{x_{i-2}-x_i}\left(2+\frac{x_i-x_{i-1}}{x_i-x_{i+1}}\right)\right\}$ $\times\left(\frac{x_{i-2}-x_{i+1}}{x_{i-2}-x_{i+2}}\right)\}+2b\{\left(\frac{x_{i-2}-x_{i+1}}{x_{i-1}-x_{i-2}}\right)\left\{\frac{x_i-x_{i+1}}{x_{i-2}-x_{i+2}}+\frac{x_i-x_{i+2}}{x_{i-2}-x_{i+2}}\left(3+\frac{x_i-x_{i+1}}{x_i-x_{i-1}}-\frac{x_{i-2}-x_{i+1}}{x_{i-2}-x_i}\right)\right\}+\left(\frac{x_{i-2}-x_{i-1}}{x_{i-2}-x_{i+2}}\right)\times$ $\left\{\frac{x_i-x_{i+1}}{x_{i-2}-x_i}+\frac{x_i-x_{i+2}}{x_{i-2}-x_i}\left(2+\frac{x_i-x_{i+1}}{x_i-x_{i-1}}\right)\right\}\}]u_{i-2}+\left\{\frac{2}{D}\right\}\left\{\frac{(x_i-x_{i+1})(x_i-x_{i-1})}{(x_{i+2}-x_{i+1})^2(x_{i+2}-x_{i-1})^2}\right\}[2\left\{\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_{i-1}}{x_{i-1}-x_{i+2}}\right\}\times$ $\left\{\frac{x_{i-1}-x_{i+1}}{x_{i+2}-x_{i-2}}\right\}\left\{\frac{x_{i-2}-x_i}{x_i-x_{i+1}}+\frac{x_{i-2}-x_i}{x_i-x_{i-1}}+\frac{2x_i-x_{i-2}-x_{i+2}}{x_{i+2}-x_i}\right\}+2a\{\left(\frac{x_{i+2}-x_{i-1}}{x_{i+1}-x_{i+2}}\right)\{\frac{x_i-x_{i-1}}{x_{i+2}-x_{i-2}}+\left(3+\frac{x_i-x_{i-1}}{x_i-x_{i+1}}-\frac{x_{i+2}-x_{i-1}}{x_{i+2}-x_i}\right)\times$ $\frac{x_i-x_{i-2}}{x_{i+2}-x_{i-2}}\}+\left\{\frac{x_i-x_{i-1}}{x_{i+2}-x_i}+\frac{x_i-x_{i-2}}{x_{i+2}-x_i}\left(2+\frac{x_i-x_{i-1}}{x_i-x_{i+1}}\right)\right\}\left(\frac{x_{i+2}-x_{i+1}}{x_{i+2}-x_{i-2}}\right)\}+2b\{\{\frac{x_i-x_{i+1}}{x_{i+2}-x_{i-2}}+\left(3+\frac{x_i-x_{i+1}}{x_i-x_{i-1}}-\frac{x_{i+2}-x_{i+1}}{x_{i+2}-x_i}\right)\times$ $\left(\frac{x_i-x_{i-2}}{x_{i+2}-x_{i-2}}\right)\}\left(\frac{x_{i+2}-x_{i+1}}{x_{i-1}-x_{i+2}}\right)+\left\{\frac{x_i-x_{i+1}}{x_{i+2}-x_i}+\frac{x_i-x_{i-2}}{x_{i+2}-x_i}\left(2+\frac{x_i-x_{i+1}}{x_i-x_{i-1}}\right)\right\}\left(\frac{x_{i+2}-x_{i-1}}{x_{i+2}-x_{i-2}}\right)\}]u_{i+2}+[\frac{2}{(x_i-x_{i+1})(x_i-x_{i-1})}+$ $\frac{2}{(x_i-x_{i+2})(x_i-x_{i-2})}+\left(\frac{1}{x_i-x_{i+1}}+\frac{1}{x_i-x_{i-1}}\right)\left(\frac{1}{x_i-x_{i+2}}+\frac{1}{x_i-x_{i-2}}\right)-\left\{ab(x_{i+1}-x_{i-1})-b\frac{x_{i+1}-x_i}{x_{i-1}-x_i}+a\frac{x_{i-1}-x_i}{x_{i+1}-x_i}\right\}\frac{4(x_{i+1}-x_{i-1})}{(x_{i+1}-x_i)(x_{i-1}-x_i)D}$ $-\frac{4(x_{i+1}-x_{i-1})}{(x_{i+1}-x_i)(x_{i-1}-x_i)D}\{-ab(x_{i+1}-x_{i-1})(x_{i+1}+x_{i-1}-2x_i)+\{(x_{i+1}-x_i)^2+(x_{i-1}-x_i)^2\}\left(\frac{b}{x_{i-1}-x_i}-\frac{a}{x_{i+1}-x_i}\right)+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_i}$ $+\frac{x_{i+1}-x_{i-1}}{x_{i-1}-x_i}\}\left\{\frac{1}{x_i-x_{i-1}}+\frac{1}{x_i-x_{i+1}}+\frac{1}{x_i-x_{i-2}}+\frac{1}{x_i-x_{i+2}}\right\}]u_i$ where $a=1+\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_i}+\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_{i+2}}$, $b=1+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_i}+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i+2}}$ and $D=6+4\left(\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_i}\right)-4\left(\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_i}\right)$ $+2\left(\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i-2}}+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_i}\right)\left(\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_{i-2}}+\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_{i+1}}{x_{i-1}-x_i}\right)\neq 0$ for scheme to exist	$\frac{2}{11}u''_{i-1}+u''_i+\frac{2}{11}u''_{i+1}=$ $\frac{12}{11h^2}(u_{i+1}-u_{i-1})+$ $\frac{3}{44h^2}(u_{i+2}-u_{i-2})$ $-\frac{51}{22h^2}u_i$

For derivation of the scheme for first derivative a choice of $\mathbf{I_n} = \{i\}$ and then a derivative of the interpolation polynomial obtained at $x = x_i$ yields

$$u'_i + \sum_{j\in I_k}\left(\tilde{a}_j u''_j + \tilde{b}_j u'_j\right) = \tilde{c}_i u_i + \sum_{j\in I_k}\tilde{c}_j u_j + \sum_{j\in I_m}\tilde{\tilde{c}}_j u_j,$$

$$\tilde{a}_j = \left(\frac{x_i - x_j}{2}\right)\frac{\prod_m(x_i)}{\prod_m(x_j)}\left\{l_j^k(x_i)\right\}^3,$$

$$\tilde{b}_j = \frac{\prod_m(x_i)}{\prod_m(x_j)}\left\{l_j^k(x_i)\right\}^3\left\{2 + (x_j - x_i)\left(\frac{\prod'_m(x_j)}{\prod_m(x_j)} + 3l_j^{k'}(x_j)\right)\right\},$$

$$\tilde{c}_i = \left\{\frac{\prod'_m(x_i)}{\prod_m(x_i)} + 3\frac{\prod'_k(x_i)}{\prod_k(x_i)}\right\},$$

Table B.3
Combined compact schemes on uniform and non-uniform stencils

Non-uniform grid	Uniform grid
$-\left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^3 u''_{i-1} + u''_i - \left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^3 u''_{i+1} + 6\left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^3\left\{\frac{1}{x_{i-1}-x_i}+\frac{1}{x_{i-1}-x_{i+1}}\right\}u'_{i-1} - 6\left\{\frac{1}{x_i-x_{i+1}}+\frac{1}{x_i-x_{i-1}}\right\}u'_i$ $+6\left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^3\left\{\frac{1}{x_{i+1}-x_i}+\frac{1}{x_{i+1}-x_{i-1}}\right\}u'_{i+1} = \left\{\frac{2}{(x_i-x_{i-1})^2}\right\}\left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^3\left\{6+9\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}+6\frac{(x_{i-1}-x_i)^2}{(x_{i-1}-x_{i+1})^2}\right\}u_{i-1}+$ $\left\{\frac{6}{(x_i-x_{i+1})(x_i-x_{i-1})} - 12\left(\frac{1}{x_i-x_{i-1}}+\frac{1}{x_i-x_{i+1}}\right)^2\right\}u_i + \left\{\frac{2}{(x_i-x_{i+1})^2}\right\}\left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^3\left\{6+9\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}+6\frac{(x_{i+1}-x_i)^2}{(x_{i+1}-x_{i-1})^2}\right\}u_{i+1}$	$-\frac{1}{8}u''_{i-1}+u''_i-\frac{1}{8}u''_{i+1}$ $+\frac{9}{8h}(u'_{i+1}-u'_{i-1}) =$ $\frac{3}{h^2}(u_{i+1}-2u_i+u_{i-1})$
$\left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^3\left\{2+3\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}\right\}u'_{i-1}+u'_i+\left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^3\left\{2+3\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}\right\}u'_{i+1}+\left(\frac{x_i-x_{i-1}}{2}\right)\left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^3u''_{i-1}$ $\left(\frac{x_i-x_{i+1}}{2}\right)\left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^3u''_{i+1} = \left\{\frac{1}{x_{i-1}-x_i}\right\}\left\{\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right\}^3\left\{3+6\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}+6\frac{(x_{i-1}-x_i)^2}{(x_{i-1}-x_{i+1})^2}\right\}u_{i-1}+$ $3\left\{\frac{1}{x_i-x_{i+1}}+\frac{1}{x_i-x_{i-1}}\right\}u_i+\left\{\frac{1}{x_{i+1}-x_i}\right\}\left\{\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right\}^3\left\{3+6\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}+6\frac{(x_{i+1}-x_i)^2}{(x_{i+1}-x_{i-1})^2}\right\}u_{i+1}$	$\frac{7}{16}u'_{i-1}+u'_i+\frac{7}{16}u'_{i+1}$ $+\frac{h}{16}(u''_{i-1}-u''_{i+1})$ $=\frac{15}{16h}(u_{i+1}-u_{i-1})$
$-\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+1})^3}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+1})^3}\right\}u''_{i-1}+u''_i-\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i-1})^3}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})(x_{i+1}-x_{i-1})^3}\right\}u''_{i+1}+\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+1})^3}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+1})^3}\right\}$ $\times\left\{\frac{2(2x_{i-1}-x_{i+2}-x_{i-2})}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})}+\frac{6(2x_{i-1}-x_{i+1}-x_i)}{(x_{i-1}-x_{i+1})(x_{i-1}-x_i)}\right\}u'_{i-1}+\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}\right\}\left\{\frac{2(2x_{i+1}-x_{i+2}-x_{i-2})}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}+\frac{6(x_{i+1}-x_{i-1}-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}\right\}$ $\times\left\{\frac{(x_i-x_{i-1})^3}{(x_{i+1}-x_{i-1})^3}\right\}u'_{i+1}-\left\{\frac{2}{x_i-x_{i+2}}+\frac{2}{x_i-x_{i-2}}+\frac{6}{x_i-xi+1}+\frac{6}{x_i-x_{i-1}}\right\}u'_i=\left\{\frac{2(x_i-x_{i+2})(x_i-x_{i-1})^3(x_i-x_{i+1})^3}{(x_i-x_{i-2})^2(x_{i-2}-x_{i+2})(x_{i-2}-x_{i-1})^3(x_{i-2}-x_{i+1})^3}\right\}u_{i-2}+$ $\left\{\frac{2(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+1})^3}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_i-x_{i-1})^2(x_{i-1}-x_{i+1})^3}\right\}\left\{6+3\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+3\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+9\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}+6\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}\right)^2+3\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}\right)\times\right.$ $\left.\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}\right)+\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}\right)^2-\frac{1}{(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+2})}\right\}u_{i-1}+\left\{\frac{2}{(x_i-x_{i-2})(x_i-x_{i+2})}+\frac{6}{(x_i-x_{i-1})(x_i-x_{i+1})}\right.$ $\left.-2\left(\frac{1}{x_i-x_{i+2}}+\frac{1}{x_i-x_{i-2}}\right)^2-12\left(\frac{1}{x_i-x_{i+1}}+\frac{1}{x_i-x_{i-1}}\right)^2-6\left(\frac{1}{x_i-x_{i+1}}+\frac{1}{x_i-x_{i-1}}\right)\left(\frac{1}{x_i-x_{i+2}}+\frac{1}{x_i-x_{i-2}}\right)\right\}u_i+\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}\right\}\times$ $\left\{6+3\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+3\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+9\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}+6\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}\right)^2+3\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}\right)\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}\right)+\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}\right)^2\right.$ $\left.-\frac{1}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})}\right\}\left\{\frac{2(x_i-x_{i-1})^3}{(x_i-x_{i+1})^2(x_{i+1}-x_{i-1})^3}\right\}u_{i+1}+\left\{\frac{2(x_i-x_{i-2})(x_i-x_{i-1})^3(x_i-x_{i+1})^3}{(x_i-x_{i+2})^2(x_{i+2}-x_{i-2})(x_{i+2}-x_{i-1})^3(x_{i+2}-x_{i+1})^3}\right\}u_{i+2}$	$-\frac{1}{6}u''_{i-1}+u''_i-\frac{1}{6}u''_{i+1}$ $+\frac{23}{18h}(u'_{i+1}-u'_{i-1})$ $=\frac{88}{27h^2}(u_{i+1}+u_{i-1})$ $-\frac{1}{108h^2}(u_{i+2}+u_{i-2})$ $-\frac{23}{4h^2}u_i$
$u'_i+\left\{\frac{(x_i-x_{i-1})(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i-1})^3}{2(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+1})^3}\right\}u''_{i-1}+\left\{\frac{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i-1})^3}{2(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})(x_{i+1}-x_{i-1})^3}\right\}u''_{i+1}+\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+1})^3}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+1})^3}\right\}$ $\times\left\{2+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+3\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}\right\}u'_{i-1}+\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i-1})^3}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})(x_{i+1}-x_{i-1})^3}\right\}\left\{2+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+3\frac{x_{i+1}-x_i}{x_{i+1}-xi-1}\right\}u'_{i+1}$ $=\left\{\frac{2x_i-x_{i+2}-x_{i-2}}{(x_i-x_{i+2})(x_i-x_{i-2})}+\frac{3(x_i-x_{i+1}-x_{i-1})}{(x_i-x_{i+1})(x_i-x_{i-1})}\right\}u_i+\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-1})^3(x_i-x_{i+1})^3}{(x_{i-2}-x_i)(x_{i-2}-x_{i+2})(x_{i-2}-x_{i-1})^3(x_{i-2}-x_{i+1})^3}\right\}u_{i-2}+\left\{3+2\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+2\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}+\right.$ $\left.6\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}+6\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}\right)^2+3\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+1}}\right)\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}\right)+\left(\frac{x_{i-1}-x_i}{x_{i-1}-x_{i+2}}+\frac{x_{i-1}-x_i}{x_{i-1}-x_{i-2}}\right)^2-\frac{1}{(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+2})}\right\}\times$ $\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i+1})^3}{(x_{i-1}-x_{i+2})(x_{i-1}-x_{i-2})(x_{i-1}-x_{i+1})^3(x_{i-1}-x_i)}\right\}u_{i-1}+\left\{3+2\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}+2\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+6\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}+6\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}\right)^2+3\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}\right)\times\right.$ $\left.\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}\right)+\left(\frac{x_{i+1}-x_i}{x_{i+1}-x_{i+2}}+\frac{x_{i+1}-x_i}{x_{i+1}-x_{i-2}}\right)^2-\frac{1}{(x_{i+1}-x_{i-2})(x_{i+1}-x_{i+2})}\right\}\left\{\frac{(x_i-x_{i+2})(x_i-x_{i-2})(x_i-x_{i-1})^3}{(x_{i+1}-x_{i+2})(x_{i+1}-x_{i-2})(x_{i+1}-x_i)(x_{i+1}-x_{i-1})^3}\right\}u_{i+1}+$ $\left\{\frac{(x_i-x_{i-2})(x_i-x_{i-1})^3(x_i-x_{i+1})^3}{(x_{i+2}-x_i)(x_{i+2}-x_{i-2})(x_{i+2}-x_{i-1})^3(x_{i+2}-x_{i+1})^3}\right\}u_{i+2}$	$\frac{17}{36}u'_{i-1}+u'_i+\frac{17}{36}u'_{i+1}$ $+\frac{h}{12}(u''_{i-1}-u''_{i+1})$ $=\frac{107}{108h}(u_{i+1}-u_{i-1})$ $-\frac{1}{108h}(u_{i+2}-u_{i-2})$

$$\tilde{\hat{c}}_j = \left\{\frac{\prod_k(x_i)}{\prod_k(x_j)}\right\}^3 \frac{l_j^m(x_i)}{x_j - x_i},$$

$$\tilde{c}_j = \frac{\left\{l_j^k(x_i)\right\}^3}{x_j - x_i}\left\{\frac{\prod_m(x_i)}{\prod_m(x_j)}\right\}\left\{3 + 2(x_j - x_i)\left(\frac{\prod'_m(x_j)}{\prod_m(x_j)} + 3l_j^{k'}(x_j)\right)\right.$$
$$\left. +(x_j - x_i)^2 \times \left[\left(\frac{\prod'_m(x_j)}{\prod_m(x_j)}\right)^2 + 6\left(l_j^{k'}(x_j)\right)^2 + 3\frac{\prod'_m(x_j)}{\prod_m(x_j)}l_j^{k'}(x_j) - \frac{\prod''_m(x_j)}{2\prod_m(x_j)} - \frac{3}{2}l_j^{k''}(x_j)\right]\right\}.$$

If $\mathbf{I_m}$ and $\mathbf{I_k}$ contain m and k points, respectively, then the order of the combined compact scheme will be $(3k + m)$. It may also be noted that a proper choice of sets of points $\mathbf{I_m}$ and $\mathbf{I_k}$ will yield boundary closure schemes. Two examples of sixth- and eighth-order schemes for the interior are shown in Table B.3. Note that sixth-order accurate non-uniform combined compact schemes have been presented in [14]. The sixth-order combined compact scheme for first derivative given in Table B.3 is the same as that in [14]. However, the second derivative non-uniform combined compact scheme presented in [14] is not sixth-order accurate but fifth order only which can be verified by deriving the truncation error as $6(k_i - 1)k_i^2h_i^6/7!$ using a simple Taylor expansion.

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